CSCE 658: Randomized Algorithms

Lecture 15

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Relevant Supplementary Material

- Chapter 29 in "Introduction to Algorithms", by Thomas H. Cormen, Charles E. Leiserson, Ronald L. Rivest, and Clifford Stein

- Chapters 5.1-5.5 in "The Design of Approximation Algorithms", by David P. Williamson and David B. Shmoys

MAX-SAT

- In the MAX-SAT problem, the input is a CNF formula $f(x_1, \ldots, x_n)$ with m clauses $C_1, \ldots., C_m$

- The goal is to assign values to $x_1, \ldots, x_n$ to maximize the number of satisfied clauses

MAX-SAT

- Suppose we assign each variable x_i a separate random TRUE/FALSE value
- For each $i \in [m]$, we have $\Pr[C_i \text{ is FALSE}] \leq 1/2$
- By a linearity of expectation, the expected number of satisfied clauses is at least $m/2$

- Random assignment gives (at least) a $\frac{1}{2}$-approximation in expectation

Derandomization of MAX-SAT

- How to get an algorithm that achieves $\frac{1}{2}$?

- Method of conditional expectation
  - Set x_1 to be the value with the higher conditional expectation
  - Random assignment is a $\frac{1}{2}$-approximation in expectation, so there is a value of x_1 that is a $\frac{1}{2}$-approximation in expectation
  - Iterate

Better Algorithm for MAX-SAT

- First suppose there is no unit clause $\overline{x_i}$ (will remove assumption later)

- Set each x_i to be TRUE with probability $p > 1/2$ independently

Better Algorithm for MAX-SAT

- Claim: The probability that any given clause is satisfied is $\min(p, 1-p^2)$

- If the clause has one literal, the probability the clause is TRUE is p
- Otherwise if the clause has a literals that are negated, b literals that are not negated, the probability the clause is TRUE is $1 - p^a(1-p)^b > 1 - p^2$ for $a + b \geq 2$ and $p > \frac{1}{2}$

Better Algorithm for MAX-SAT

- Claim: The probability that any given clause is satisfied is $\min(p, 1 - p^2)$
- $\min(p, 1 - p^2)$ is maximized ≈ 0.618 for $p = \frac{1}{2}(\sqrt{5} - 1)$
- If there is no unit clause $\overline{x_i}$, there is a ≈ 0.618 approximation algorithm for MAX-SAT

Better Algorithm for MAX-SAT

- Let U be the set of clauses excluding negated unit clauses $\overline{x_i}$
- Assume without loss of generality that for each fixed $i \in [n]$, the number of unit clauses x_i is at least the number of unit clauses $\overline{x_i}$
- Let v_i be the number of unit clauses $\overline{x_i}$

- $\mathrm{OPT} \leq m - \sum_{i \in [n]} v_i$ since x_i cannot be both TRUE and FALSE (and the assumption that x_i appears more than $\overline{x_i}$)

Better Algorithm for MAX-SAT

$$\sum_{j \in [m]} \Pr[C_j \text{ is satisfied}] = \sum_{j \in U} \Pr[C_j \text{ is satisfied}]$$

(each clause satisfied with probability p)

$$\geq p|U|$$

(U is the set of clauses excluding negated unit clauses $\overline{x_i}$)

$$= p\left(m - \sum_{i \in [n]} v_i\right)$$

$$\geq p \cdot OPT$$

Political Problems…

- As a politician seeking approval ratings, you would like the support of 50 urban voters, 100 suburban voters, 25 rural voters
- For each \$1 spent advertising one of the following policies, the resulting effects are:
- Optimize your budget
- (Warm-up from CLRS)

Policy	Urban	Suburban	Rural
Zombie apocalypse	-2	+5	-3
Sharks with lasers	+8	+2	-5
Flying cars roads	0	0	+10
Dolphins voting	+10	0	-2

Political Problems…

- You seek 50 urban voters, 100 suburban voters, 25 rural voters

Policy	Urban	Suburban	Rural
Zombie apocalypse	-2	+5	-3
Sharks with lasers	+8	+2	-5
Flying cars roads	0	0	+10
Dolphins voting	+10	0	-2

- Let x_1 be the money spent on ads for preparing for a zombie apocalypse, x_2 be the money for ads for sharks with lasers, x_3 be the money spent on ads for roads for flying cars, and x_4 be the money spent on ads for allowing dolphins to vote
- Urban voters: $-2x_1 + 8x_2 + 10x_4$
- Constraint: $-2x_1 + 8x_2 + 10x_4 \geq 50$

Political Problems…

Policy	Urban	Suburban	Rural
Zombie apocalypse	-2	+5	-3
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Dolphins voting	+10	0	-2

- You seek 50 urban voters, 100 suburban voters, 25 rural voters
- Let x_1 be the money spent on ads for preparing for a zombie apocalypse, x_2 be the money for ads for sharks with lasers, x_3 be the money spent on ads for roads for flying cars, and x_4 be the money spent on ads for allowing dolphins to vote
- Suburban voters: $5x_1 + 2x_2$
- Constraint: $5x_1 + 2x_2 \geq 100$

Political Problems…

Policy	Urban	Suburban	Rural
Zombie apocalypse	-2	+5	-3
Sharks with lasers	+8	+2	-5
Flying cars roads	0	0	+10
Dolphins voting	+10	0	-2

- You seek 50 urban voters, 100 suburban voters, 25 rural voters
- Let x_1 be the money spent on ads for preparing for a zombie apocalypse, x_2 be the money for ads for sharks with lasers, x_3 be the money spent on ads for roads for flying cars, and x_4 be the money spent on ads for allowing dolphins to vote
- Rural voters: $-3x_1 - 5x_2 + 10x_3 - 2x_4$
- Constraint: $-3x_1 - 5x_2 + 10x_3 - 2x_4 \geq 25$

Minimization Problem

Policy	Urban	Suburban	Rural
Zombie apocalypse	-2	+5	-3
Sharks with lasers	+8	+2	-5
Flying cars roads	0	0	+10
Dolphins voting	+10	0	-2

- You seek 50 urban voters, 100 suburban voters, 25 rural voters
- Minimize: $x_1 + x_2 + x_3 + x_4$
- Constraints:

$$-2x_1 + 8x_2 + 10x_4 \geq 50$$
$$5x_1 + 2x_2 \geq 100$$
$$-3x_1 - 5x_2 + 10x_3 - 2x_4 \geq 25$$

Linear Programming

- Maximize a linear objective function:

$$f(x_1, \dots, x_n) = c_1 x_1 + \cdots + c_n x_n$$

- Subject to constraints:

$$\sum_{j=1}^{n} a_{i,j} x_j \le b_i \text{ for } i = 1, \dots, m$$
$$x_j \ge 0 \text{ for } j = 1, \dots, n$$

Linear Programming (Standard Form)

- Maximize a linear objective function:

$$c^\top x = \langle c, x \rangle, \;\; c, x \in \mathbb{R}^n$$

- Subject to constraints:

$$Ax \leq b \text{ for } A \in \mathbb{R}^{m \times n}, b \in \mathbb{R}^m$$

$$x \geq 0 \text{ (entry-wise non-negativity)}$$

Linear Programming (Standard Form)

- If a particular solution $\bar{x}$ satisfies all the constraints, we call it a *feasible solution*; otherwise, we call it an *infeasible solution*

- Can convert any linear program into standard form, even if there are equality constraints or variables that can take on negative values

Linear Programming

- Minimize: $x_1 + x_2$
- Subject to:

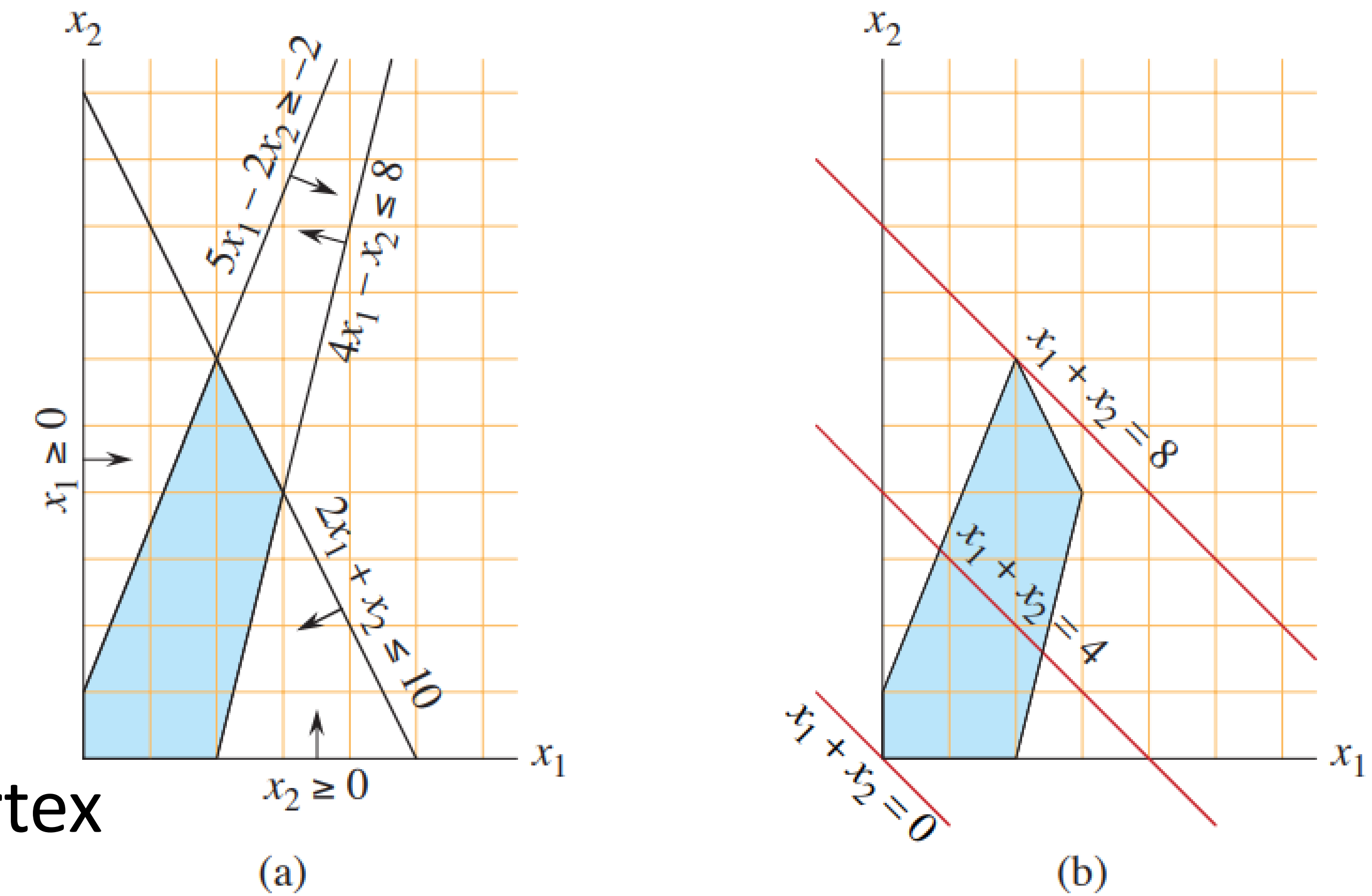
$$4x_1 - x_2 \leq 8$$
$$2x_1 + x_2 \leq 10$$
$$5x_1 - 2x_2 \leq -2$$
$$x_1, x_2 \geq 0$$

Linear Programming

- Minimize: $x_1 + x_2$
- Subject to:

$$4x_1 - x_2 \le 8$$
$$2x_1 + x_2 \le 10$$
$$5x_1 - 2x_2 \le -2$$
$$x_1, x_2 \ge 0$$

- Optimal solution is always located at vertex of feasible region



Linear Programming

- Simplex algorithm: finds a feasible solution at a vertex of the polytope and then searches along the edges to vertices with non-decreasing values

- Good in practice but exponential time in the worst-case

Linear Programming

- Ellipsoid algorithm: iterative algorithm that generates a sequence of smaller ellipsoids, each of which separate the current iterate with the optimal solution

- Requires a separation oracle

- Polynomial time algorithm in theory but inefficient in practice and can suffer from numerical instability

Duality

- Maximize $c_1x_1 + \cdots + c_nx_n$ subject to $\sum_{j=1}^{n} a_{i,j}x_j \leq b_i$ for $i = 1, \ldots, m$ and $x_j \geq 0$ for $j = 1, \ldots, n$

- For the above linear program, its dual is

  Minimize: $b_1y_1 + \cdots + b_my_m$

  Subject to: $\sum_{i=1}^{m} a_{i,j}y_i \geq c_j$ for $j = 1, \ldots, n$

  $y_i \geq 0$ for $i = 1, \ldots, m$

Duality

Maximize: $3x_1 + x_2 + 4x_3$

Subject to: $x_1 + x_2 + 3x_3 \leq 30$

$$2x_1 + 2x_2 + 5x_3 \leq 24$$

$$4x_1 + x_2 + 2x_3 \leq 36$$

$$x_1, x_2, x_3 \geq 0$$

- What is its dual program? (Williamson-Shmoys)

Duality

Minimize: $30y_1 + 24y_2 + 36y_3$

Subject to: $y_1 + 2y_2 + 4y_3 \geq 3$

$$y_1 + 2y_2 + y_3 \geq 1$$

$$3y_1 + 5y_2 + 2y_3 \geq 4$$

$$y_1, y_2, y_3 \geq 0$$

Duality

- Intuition: What if we take the original constraints and add the first two constraints?

Maximize: $3x_1 + x_2 + 4x_3$

Subject to: $x_1 + x_2 + 3x_3 \leq 30$

$$2x_1 + 2x_2 + 5x_3 \leq 24$$

$$4x_1 + x_2 + 2x_3 \leq 36$$

$$x_1, x_2, x_3 \geq 0$$

Duality

- Intuition: What if we take the original constraints and add the first two constraints?

$$3x_1 + 3x_2 + 8x_3 \leq 54$$

Maximize: $3x_1 + x_2 + 4x_3$

Subject to: $x_1 + x_2 + 3x_3 \leq 30$

$$2x_1 + 2x_2 + 5x_3 \leq 24$$

$$4x_1 + x_2 + 2x_3 \leq 36$$

$$x_1, x_2, x_3 \geq 0$$

- Coefficients all larger than objective
- The primal solution must be at most 54

Duality

- Intuition: What if we take the original constraints and add the first two constraints?

Maximize: $3x_1 + x_2 + 4x_3$

Subject to: $x_1 + x_2 + 3x_3 \leq 30$

$$2x_1 + 2x_2 + 5x_3 \leq 24$$

$$4x_1 + x_2 + 2x_3 \leq 36$$

$$x_1, x_2, x_3 \geq 0$$

- Can we find a linear combination of the equations that exactly matches the objective?

Duality

- Intuition: What if we take the original constraints and add the first two constraints?

Maximize: $3x_1 + x_2 + 4x_3$

Subject to: $x_1 + x_2 + 3x_3 \leq 30$

$$2x_1 + 2x_2 + 5x_3 \leq 24$$

$$4x_1 + x_2 + 2x_3 \leq 36$$

$$x_1, x_2, x_3 \geq 0$$

- Suppose we multiply the first constraint by y_1, the second constraint by y_2, the third constraint by y_3
- What relationship do we get for x_1?

Duality

Minimize: $30y_1 + 24y_2 + 36y_3$

Subject to: $y_1 + 2y_2 + 4y_3 \geq 3$

$$y_1 + 2y_2 + y_3 \geq 1$$

$$3y_1 + 5y_2 + 2y_3 \geq 4$$

$$y_1, y_2, y_3 \geq 0$$

- The dual is exactly this program

Duality

- Weak duality: any feasible solution to the primal linear program has objective at most any feasible solution to the dual linear program, i.e., $\langle c, \overline{x} \rangle \leq \langle b, \overline{y} \rangle$

- LP duality: If both the primal linear program and the corresponding dual are feasible and bounded, then for optimal solutions x^* and y^*, $\langle c, x^* \rangle = \langle b, y^* \rangle$