# CSCE 658: Randomized Algorithms

Lecture 9

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# Homework 1

- Quick overview of the solutions

# Last Time: CountMin

- Initalization: Create $b$ buckets of counters and use a random hash function $h: [n] \to [b]$
- Algorithm: For each update $x_i$, increment the counter $h(x_i)$

| $c_1$ | $c_2$ | $c_3$ | $c_4$ |
|---|---|---|---|
| 0 | 0 | 0 | 0 |

- At the end of the stream, output the counter $h(x_i)$ as the estimate for $x_i$

# CountMin for $(\varepsilon, k)$-Frequent Items Problem

- Claim: For all estimated frequencies $\widehat{f_i}$ by CountMin, we have

$$f_i - \frac{\varepsilon \|f\|_1}{3k} \le \widehat{f_i} \le f_i + \frac{\varepsilon \|f\|_1}{3k}$$

- If $f_i \ge \frac{\|f\|_1}{k}$, then $\widehat{f_i} \ge f_i - \frac{\varepsilon \|f\|_1}{2k}$ and if $f_i < (1-\varepsilon) \cdot \frac{\|f\|_1}{k}$, then $\widehat{f_i} < f_i - \frac{\varepsilon \|f\|_1}{2k}$
- Returning coordinates $V_t$ with $c_t \ge \left(1 - \frac{\varepsilon}{2}\right) \cdot \frac{\|f\|_1}{k}$ means:
  - $i$ with $f_i \ge \frac{\|f\|_1}{k}$ will be returned
  - NO $i$ with $f_i < (1-\varepsilon) \cdot \frac{\|f\|_1}{k}$ will be returned

# CountMin for $(\varepsilon, k)$-Frequent Items Problem

- Summary: CountMin can be used to solve the $(\varepsilon, k)$-frequent items problem on an insertion-deletion stream

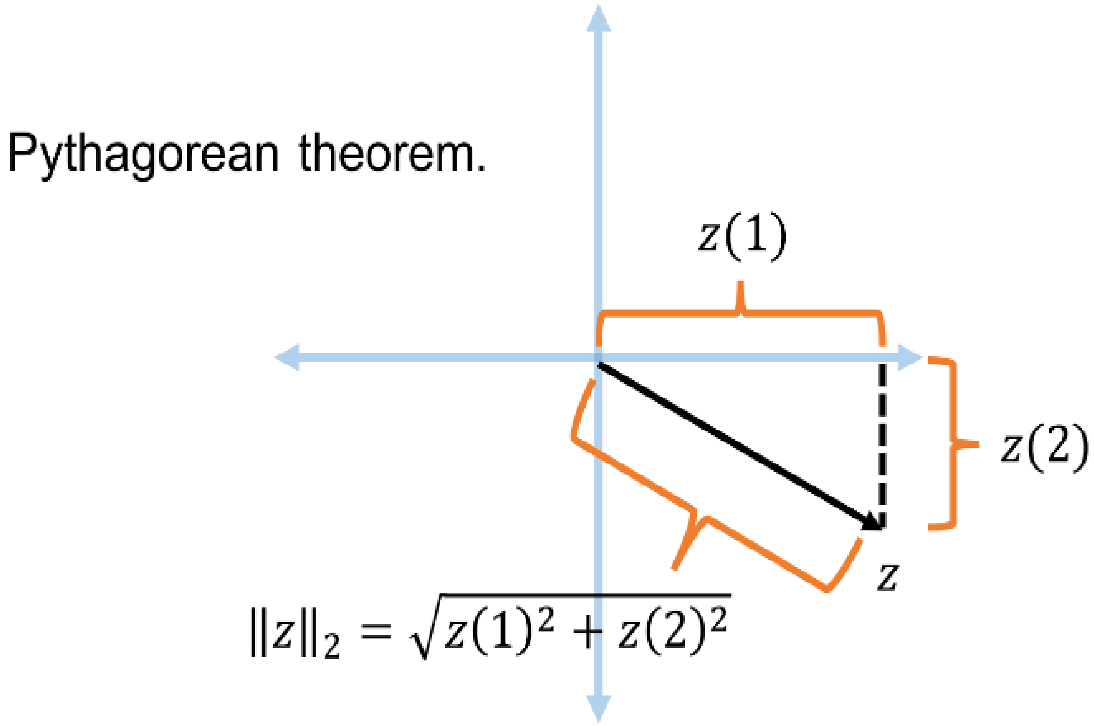
- CountMin uses $O\left(\frac{k}{\varepsilon}\log^2 n\right)$ bits of space

- CountMin is a randomized algorithm

# Recall: Euclidean Space and $L_2$ Norm

- For $z \in R^n$, the $L_2$ norm of $z$ is denoted by $\|z\|_2$ and defined as:

$$\|z\|_2 = \sqrt{z_1^2 + z_2^2 + \cdots + z_n^2}$$




- For $x, y \in R^n$, the distance function $D$ is denoted by $\|\cdot\|_2$ and defined as $\|x - y\|_2$

# Trivia Question #7 (Norms)

- For $x \in R^n$, which of the following is (the most) true?

- $\|x\|_2 > \|x\|_1$
- $\|x\|_2 \geq \|x\|_1$
- $\|x\|_2 = \|x\|_1$
- $\|x\|_2 \leq \|x\|_1$
- $\|x\|_2 < \|x\|_1$
- None of these are true characterizations of the relationship between $\|x\|_2$ and $\|x\|_1$

# Trivia Question #8 (Norms)

- For $x \in R^n$, how much large can $\|x\|_1 / \|x\|_2$ be?

- $O(n)$
- $O(\sqrt{n})$
- $O(\log n)$
- $O(1)$

# $(\varepsilon, k)$-Frequent Items Problem

- Goal: Given a set $S$ of $m$ elements from $[n]$ that induces a frequency vector $f \in R^n$, an accuracy parameter $\varepsilon \in (0,1)$, and a parameter $k$, output a list that includes:
  - The items from $[n]$ that have frequency at least $\frac{\|f\|_1}{k}$
  - No items with frequency less than $(1-\varepsilon)\frac{\|f\|_1}{k}$

# $L_2$ Heavy-Hitters

- Goal: Given a set $S$ of $m$ elements from $[n]$ that induces a frequency vector $f \in R^n$, an accuracy parameter $\varepsilon \in (0,1)$, and a parameter $k$, output a list that includes:
  - The items from $[n]$ that have frequency at least $\frac{\|f\|_2}{k}$
  - No items with frequency less than $(1-\varepsilon)\frac{\|f\|_2}{k}$

# $L_2$ Heavy-Hitters

- Goal: Given a set $S$ of $m$ elements from $[n]$ that induces a frequency vector $f \in R^n$ and a threshold parameter $\varepsilon \in (0, 1)$, output a list that includes:
  - The items from $[n]$ that have frequency at least $\varepsilon \cdot ||f||_2$
  - No items with frequency less than $\frac{\varepsilon}{2} \cdot ||f||_2$

# $L_2$ Estimation

- Goal: Given a set $S$ of $m$ elements from $[n]$ that induces a frequency vector $f \in R^n$ and an accuracy parameter $\varepsilon \in (0, 1)$, output a $(1+\varepsilon)$-approximation to $\|f\|_2$

- Find $Z$ such that $(1-\varepsilon) \cdot \|f\|_2 \leq Z \leq (1+\varepsilon) \cdot \|f\|_2$

- Find $Z'$ such that $(1-\varepsilon) \cdot \|f\|_2^2 \leq Z' \leq (1+\varepsilon) \cdot \|f\|_2^2$

# $L_2$ Estimation

- How to do?

# $L_2$ Estimation

- How to do?
- Recall: Johnson-Lindenstrauss Transformation
- Assume for now we are given $\|f\|_2$

# Revisiting CountMin

- Initalization: Create $b$ buckets of counters and use a random hash function $h: [n] \to [b]$
- Algorithm: For each insertion (or deletion) to $x_i$, increment (or decrement) the counter $h(x_i)$

| $c_1$ | $c_2$ | $c_3$ | $c_4$ |
|---|---|---|---|
| 0 | 0 | 0 | 0 |

- At the end of the stream, output the counter $h(x_i)$ as the estimate for $x_i$

# CountMin and the Power of Random Signs

- Initalization: Create $b$ buckets of counters and use a random hash function $h: [n] \rightarrow [b]$ and a uniformly random sign function $s: [n] \rightarrow \{-1, +1\}$, i.e., $\Pr[s(i) = +1] = \Pr[s(i) = -1] = \frac{1}{2}$
- Algorithm: For each insertion (or deletion) to $x_i$, change the counter $h(x_i)$ by $s(x_i)$ (or $-s(x_i)$)

| $c_1$ | $c_2$ | $c_3$ | $c_4$ |
|---|---|---|---|
| 0 | 0 | 0 | 0 |

- At the end of the stream, output the quantity $s(x_i) \cdot h(x_i)$ as the estimate for $x_i$

# CountSketch

| $f_1$ | $f_2$ | $f_3$ | $f_4$ | $f_5$ | $f_6$ | $f_7$ |
|---|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 | 0 | 0 |

1

| $c_1$ | $c_2$ | $c_3$ | $c_4$ |
|---|---|---|---|
| 0 | 0 | 0 | 0 |

# CountSketch

| $f_1$ | $f_2$ | $f_3$ | $f_4$ | $f_5$ | $f_6$ | $f_7$ |
|---|---|---|---|---|---|---|
| 1 | 0 | 0 | 0 | 0 | 0 | 0 |

$h(x) = 3x + 2 \pmod 4$
$s(x) = +1$ for $x \in \{1,2,3,6,7\}$
$s(x) = -1$ for $x \in \{4,5\}$

1

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|---|---|---|---|
| 0 | 0 | 0 | 0 |

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$1$

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| $f_1$ | $f_2$ | $f_3$ | $f_4$ | $f_5$ | $f_6$ | $f_7$ |
|---|---|---|---|---|---|---|
| 1 | 0 | 0 | 0 | 0 | 0 | 0 |

3

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|---|---|---|---|
| 1 | 0 | 0 | 0 |

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| $f_1$ | $f_2$ | $f_3$ | $f_4$ | $f_5$ | $f_6$ | $f_7$ |
|---|---|---|---|---|---|---|
| 1 | 0 | 1 | 0 | 0 | 0 | 0 |

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|---|---|---|---|
| 1 | 0 | 1 | 0 |

# CountSketch

| $f_1$ | $f_2$ | $f_3$ | $f_4$ | $f_5$ | $f_6$ | $f_7$ |
|---|---|---|---|---|---|---|
| 1 | 0 | 1 | 0 | 0 | 0 | 0 |

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$s(x) = -1$ for $x \in \{4,5\}$

$$2$$

| $c_1$ | $c_2$ | $c_3$ | $c_4$ |
|---|---|---|---|
| 1 | 0 | 1 | 0 |

# CountSketch

| $f_1$ | $f_2$ | $f_3$ | $f_4$ | $f_5$ | $f_6$ | $f_7$ |
|---|---|---|---|---|---|---|
| 1 | 1 | 1 | 0 | 0 | 0 | 0 |

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|---|---|---|---|
| 1 | 0 | 1 | 1 |

# CountSketch

| $f_1$ | $f_2$ | $f_3$ | $f_4$ | $f_5$ | $f_6$ | $f_7$ |
|---|---|---|---|---|---|---|
| 1 | 1 | 1 | 0 | 0 | 0 | 0 |

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|---|---|---|---|
| 1 | 0 | 1 | 1 |

# CountSketch

| $f_1$ | $f_2$ | $f_3$ | $f_4$ | $f_5$ | $f_6$ | $f_7$ |
|---|---|---|---|---|---|---|
| 2 | 1 | 1 | 0 | 0 | 0 | 0 |

$h(x) = 3x + 2 \pmod 4$
$s(x) = +1$ for $x \in \{1,2,3,6,7\}$
$s(x) = -1$ for $x \in \{4,5\}$

1

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|---|---|---|---|
| 2 | 0 | 1 | 1 |

# CountSketch

| $f_1$ | $f_2$ | $f_3$ | $f_4$ | $f_5$ | $f_6$ | $f_7$ |
| --- | --- | --- | --- | --- | --- | --- |
| 2 | 1 | 1 | 0 | 0 | 0 | 0 |

5

$h(x) = 3x + 2 \pmod 4$

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$s(x) = -1$ for $x \in \{4,5\}$

| $c_1$ | $c_2$ | $c_3$ | $c_4$ |
| --- | --- | --- | --- |
| 2 | 0 | 1 | 1 |

# CountSketch

| $f_1$ | $f_2$ | $f_3$ | $f_4$ | $f_5$ | $f_6$ | $f_7$ |
|---|---|---|---|---|---|---|
| 2 | 1 | 1 | 0 | 1 | 0 | 0 |

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|---|---|---|---|
| 1 | 0 | 1 | 1 |

# CountSketch

| $f_1$ | $f_2$ | $f_3$ | $f_4$ | $f_5$ | $f_6$ | $f_7$ |
|---|---|---|---|---|---|---|
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$h(x) = 3x + 2 \pmod 4$

$s(x) = +1$ for $x \in \{1,2,3,6,7\}$

$s(x) = -1$ for $x \in \{4,5\}$

- What is the estimation for $f_4$?
- What about $f_3$?
- What about $f_5$? What about $f_1$?

| $c_1$ | $c_2$ | $c_3$ | $c_4$ |
|---|---|---|---|
| 1 | 0 | 1 | 1 |

# CountSketch

- Given a set $S$ of $m$ elements from $[n]$, let $\widehat{f_i}$ be the estimated frequency for $f_i$

- Suppose $h(i) = a$ so that $\widehat{f_i} = s(i) \cdot c_a$
- Note that $c_a$ includes the signed number $s(j) \cdot f_j$ of occurrences of any $j$ with $h(j) = a = h(i)$, including $f_i$ itself

# CountSketch

- Suppose $h(i) = a$ so that $c_a = \widehat{f_i}$
- Note that $c_a$ includes the signed number $s(j) \cdot f_j$ of occurrences of any $j$ with $h(j) = a = h(i)$, including $f_i$ itself

- $c_a = \sum_{j:h(j)=a} s(j) \cdot f_a$
- Estimated frequency $f_i$ of $i$ is $\widehat{f_i} = s(i) \cdot c_a$
- $s(i)\, c_a = s(i) \cdot s(i) \cdot f_i + \sum_{j \neq i, \text{ with } j:h(j)=a} s(i) \cdot s(j) \cdot f_j$