# CSCE 689: Special Topics in Modern Algorithms for Data Science

## Lecture 12

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# Presentation Schedule

- September 25: Team DAP, Team Bokun, Team Jason
- September 27: Galaxy AI, Team STMI
- September 29: Jung, Anmol, Chunkai

# Last Time: $(\varepsilon, k)$-Frequent Items Problem

- Goal: Given a set $S$ of $m$ elements from $[n]$, an accuracy parameter $\varepsilon \in (0,1)$, and a parameter $k$, output a list that includes:

  - The items from $[n]$ that have frequency at least $\frac{m}{k}$

  - No items with frequency less than $(1 - \varepsilon)\frac{m}{k}$

# $(\varepsilon, k)$-Frequent Items Problem

- Goal: Given a set $S$ of $m$ elements from $[n]$ that induces a frequency vector $f \in R^n$, an accuracy parameter $\varepsilon \in (0,1)$, and a parameter $k$, output a list that includes:

  - The items from $[n]$ that have frequency at least $\frac{\|f\|_1}{k}$

  - No items with frequency less than $(1 - \varepsilon)\frac{\|f\|_1}{k}$

# CountMin

- Initalization: Create $b$ buckets of counters and use a random hash function $h: [n] \to [b]$

- Algorithm: For each update $x_i$, increment the counter $h(x_i)$

| $c_1$ | $c_2$ | $c_3$ | $c_4$ |
|:-----:|:-----:|:-----:|:-----:|
| 0 | 0 | 0 | 0 |

- At the end of the stream, output the counter $h(x_i)$ as the estimate for $x_i$

# CountMin for $(\varepsilon, k)$-Frequent Items Problem

- Claim: For all estimated frequencies $\widehat{f_i}$ by CountMin, we have

$$f_i - \frac{\varepsilon \|f\|_1}{k} \leq \widehat{f_i} \leq f_i + \frac{\varepsilon \|f\|_1}{k}$$

- If $f_i \geq \frac{\|f\|_1}{k}$, then $\widehat{f_i} \geq f_i - \frac{\varepsilon \|f\|_1}{k}$ and if $f_i < (1 - \varepsilon) \cdot \frac{\|f\|_1}{k}$, then $\widehat{f_i} < f_i - \frac{\varepsilon \|f\|_1}{k}$

- Returning coordinates $V_t$ with $c_t \geq (1 - \varepsilon) \cdot \frac{\|f\|_1}{k}$ means:

  - $i$ with $f_i \geq \frac{\|f\|_1}{k}$ will be returned

  - NO $i$ with $f_i < (1 - \varepsilon) \cdot \frac{\|f\|_1}{k}$ will be returned

# CountMin for $(\varepsilon, k)$-Frequent Items Problem

- Summary: CountMin can be used to solve the $(\varepsilon, k)$-frequent items problem on an insertion-deletion stream

- CountMin uses $O\left(\dfrac{k}{\varepsilon} \log^2 n\right)$ bits of space

- CountMin is a randomized algorithm

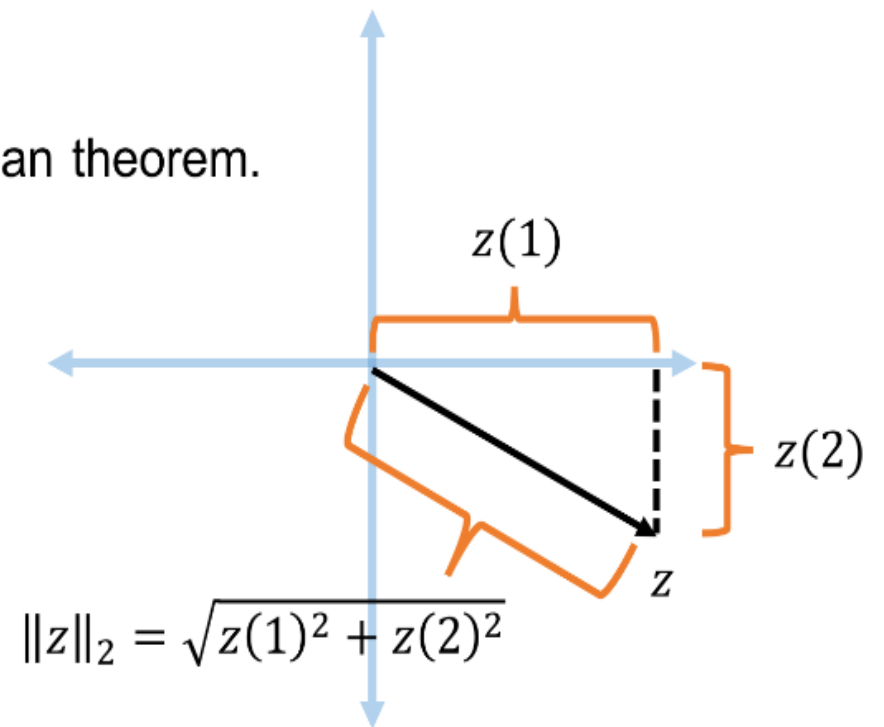- CountMin *never* underestimates the true frequency for insertion-only streams

# Recall: Euclidean Space and $L_2$ Norm

- For $z \in R^n$, the $L_2$ norm of $z$ is denoted by $\|z\|_2$ and defined as:

$$\|z\|_2 = \sqrt{z_1^2 + z_2^2 + \cdots + z_n^2}$$

Pythagorean theorem.

- For $x, y \in R^n$, the distance function $D$ is denoted by $\|\cdot\|_2$ and defined as $\|x - y\|_2$

$z(1)$

$z(2)$

$z$

$$\|z\|_2 = \sqrt{z(1)^2 + z(2)^2}$$

# Trivia Question #7 (Norms)

- For $x \in R^n$, which of the following is (the most) true?

- $\|x\|_2 > \|x\|_1$
- $\|x\|_2 \geq \|x\|_1$
- $\|x\|_2 = \|x\|_1$
- $\|x\|_2 \leq \|x\|_1$
- $\|x\|_2 < \|x\|_1$
- None of these are true characterizations of the relationship between $\|x\|_2$ and $\|x\|_1$

# Trivia Question #8 (Norms)

- For $x \in R^n$, how much large can $\|x\|_1 / \|x\|_2$ be?

- $O(n)$
- $O(\sqrt{n})$
- $O(\log n)$
- $O(1)$

# $(\varepsilon, k)$-Frequent Items Problem

- Goal: Given a set $S$ of $m$ elements from $[n]$ that induces a frequency vector $f \in R^n$, an accuracy parameter $\varepsilon \in (0, 1)$, and a parameter $k$, output a list that includes:

  - The items from $[n]$ that have frequency at least $\frac{\|f\|_1}{k}$

  - No items with frequency less than $(1 - \varepsilon)\frac{\|f\|_1}{k}$

# $L_2$ Heavy-Hitters

- Goal: Given a set $S$ of $m$ elements from $[n]$ that induces a frequency vector $f \in R^n$, an accuracy parameter $\varepsilon \in (0, 1)$, and a parameter $k$, output a list that includes:

  - The items from $[n]$ that have frequency at least $\frac{\|f\|_2}{k}$

  - No items with frequency less than $(1 - \varepsilon)\frac{\|f\|_2}{k}$

# $L_2$ Heavy-Hitters

- Goal: Given a set $S$ of $m$ elements from $[n]$ that induces a frequency vector $f \in R^n$ and a threshold parameter $\varepsilon \in (0,1)$, output a list that includes:

  - The items from $[n]$ that have frequency at least $\varepsilon \cdot \|f\|_2$

  - No items with frequency less than $\frac{\varepsilon}{2} \cdot \|f\|_2$

# $L_2$ Estimation

- Goal: Given a set $S$ of $m$ elements from $[n]$ that induces a frequency vector $f \in R^n$ and an accuracy parameter $\varepsilon \in (0, 1)$, output a $(1 + \varepsilon)$-approximation to $\|f\|_2$

- Find $Z$ such that $(1 - \varepsilon) \cdot \|f\|_2 \leq Z \leq (1 + \varepsilon) \cdot \|f\|_2$

- Find $Z'$ such that $(1 - \varepsilon) \cdot \|f\|_2^2 \leq Z' \leq (1 + \varepsilon) \cdot \|f\|_2^2$

# $L_2$ Estimation

- How to do?

# $L_2$ Estimation

- How to do?

- Sorry, won't reveal until next lecture (wait, don't we already have a tool for this)?

- Assume for now we are given $\|f\|_2$

# Revisiting CountMin

- Initalization: Create $b$ buckets of counters and use a random hash function $h: [n] \rightarrow [b]$

- Algorithm: For each insertion (or deletion) to $x_i$, increment (or decrement) the counter $h(x_i)$

| $c_1$ | $c_2$ | $c_3$ | $c_4$ |
|-------|-------|-------|-------|
| 0 | 0 | 0 | 0 |

- At the end of the stream, output the counter $h(x_i)$ as the estimate for $x_i$

# CountMin and the Power of Random Signs

- Initalization: Create $b$ buckets of counters and use a random hash function $h: [n] \to [b]$ and a uniformly random sign function $s: [n] \to \{-1, +1\}$, i.e., $\Pr[s(i) = +1] = \Pr[s(i) = -1] = \frac{1}{2}$

- Algorithm: For each insertion (or deletion) to $x_i$, change the counter $h(x_i)$ by $s(x_i)$ (or $-s(x_i)$)

| $c_1$ | $c_2$ | $c_3$ | $c_4$ |
|-------|-------|-------|-------|
| 0 | 0 | 0 | 0 |

- At the end of the stream, output the quantity $s(x_i) \cdot h(x_i)$ as the estimate for $x_i$

# CountSketch

| $f_1$ | $f_2$ | $f_3$ | $f_4$ | $f_5$ | $f_6$ | $f_7$ |
|-------|-------|-------|-------|-------|-------|-------|
| 0 | 0 | 0 | 0 | 0 | 0 | 0 |

# 1

| $c_1$ | $c_2$ | $c_3$ | $c_4$ |
|-------|-------|-------|-------|
| 0 | 0 | 0 | 0 |

# CountSketch

| $f_1$ | $f_2$ | $f_3$ | $f_4$ | $f_5$ | $f_6$ | $f_7$ |
|-------|-------|-------|-------|-------|-------|-------|
| 1     | 0     | 0     | 0     | 0     | 0     | 0     |

1

$h(x) = 3x + 2 \ (\text{mod } 4)$
$s(x) = +1$ for $x \in \{1,2,3\}$
$s(x) = -1$ for $x \in \{4,5\}$

| $c_1$ | $c_2$ | $c_3$ | $c_4$ |
|-------|-------|-------|-------|
| 0     | 0     | 0     | 0     |

# CountSketch

| $f_1$ | $f_2$ | $f_3$ | $f_4$ | $f_5$ | $f_6$ | $f_7$ |
|-------|-------|-------|-------|-------|-------|-------|
| 1 | 0 | 0 | 0 | 0 | 0 | 0 |

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**1**

| $c_1$ | $c_2$ | $c_3$ | $c_4$ |
|-------|-------|-------|-------|
| 0 | 0 | 0 | 0 |

# CountSketch

| $f_1$ | $f_2$ | $f_3$ | $f_4$ | $f_5$ | $f_6$ | $f_7$ |
|-------|-------|-------|-------|-------|-------|-------|
| 1 | 0 | 0 | 0 | 0 | 0 | 0 |

$h(x) = 3x + 2 \pmod 4$
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$s(x) = -1$ for $x \in \{4,5\}$

$1$

| $c_1$ | $c_2$ | $c_3$ | $c_4$ |
|-------|-------|-------|-------|
| 1 | 0 | 0 | 0 |

# CountSketch

| $f_1$ | $f_2$ | $f_3$ | $f_4$ | $f_5$ | $f_6$ | $f_7$ |
|-------|-------|-------|-------|-------|-------|-------|
| 1     | 0     | 0     | 0     | 0     | 0     | 0     |

3

$h(x) = 3x + 2 \ (\text{mod } 4)$
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| $c_1$ | $c_2$ | $c_3$ | $c_4$ |
|-------|-------|-------|-------|
| 1     | 0     | 0     | 0     |

# CountSketch

| $f_1$ | $f_2$ | $f_3$ | $f_4$ | $f_5$ | $f_6$ | $f_7$ |
|-------|-------|-------|-------|-------|-------|-------|
| 1 | 0 | 1 | 0 | 0 | 0 | 0 |

$h(x) = 3x + 2 \ (\text{mod } 4)$
$s(x) = +1$ for $x \in \{1,2,3\}$
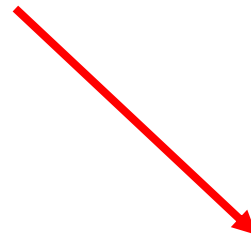$s(x) = -1$ for $x \in \{4,5\}$

3

| $c_1$ | $c_2$ | $c_3$ | $c_4$ |
|-------|-------|-------|-------|
| 1 | 0 | 1 | 0 |

# CountSketch

| $f_1$ | $f_2$ | $f_3$ | $f_4$ | $f_5$ | $f_6$ | $f_7$ |
|:---:|:---:|:---:|:---:|:---:|:---:|:---:|
| 1 | 0 | 1 | 0 | 0 | 0 | 0 |

$$2$$

$h(x) = 3x + 2 \ (\mathrm{mod}\ 4)$

$s(x) = +1$ for $x \in \{1,2,3\}$

$s(x) = -1$ for $x \in \{4,5\}$

| $c_1$ | $c_2$ | $c_3$ | $c_4$ |
|:---:|:---:|:---:|:---:|
| 1 | 0 | 1 | 0 |

# CountSketch

| $f_1$ | $f_2$ | $f_3$ | $f_4$ | $f_5$ | $f_6$ | $f_7$ |
|-------|-------|-------|-------|-------|-------|-------|
| 1 | 1 | 1 | 0 | 0 | 0 | 0 |

$h(x) = 3x + 2 \ (\text{mod } 4)$

$s(x) = +1$ for $x \in \{1,2,3\}$

$s(x) = -1$ for $x \in \{4,5\}$

2

| $c_1$ | $c_2$ | $c_3$ | $c_4$ |
|-------|-------|-------|-------|
| 1 | 0 | 1 | 1 |

# CountSketch

| $f_1$ | $f_2$ | $f_3$ | $f_4$ | $f_5$ | $f_6$ | $f_7$ |
|-------|-------|-------|-------|-------|-------|-------|
| 1 | 1 | 1 | 0 | 0 | 0 | 0 |

# 1

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| $c_1$ | $c_2$ | $c_3$ | $c_4$ |
|-------|-------|-------|-------|
| 1 | 0 | 1 | 1 |

# CountSketch

| $f_1$ | $f_2$ | $f_3$ | $f_4$ | $f_5$ | $f_6$ | $f_7$ |
|-------|-------|-------|-------|-------|-------|-------|
| 2 | 1 | 1 | 0 | 0 | 0 | 0 |

$h(x) = 3x + 2 \ (\text{mod } 4)$
$s(x) = +1 \text{ for } x \in \{1,2,3\}$
$s(x) = -1 \text{ for } x \in \{4,5\}$

**1**

| $c_1$ | $c_2$ | $c_3$ | $c_4$ |
|-------|-------|-------|-------|
| 2 | 0 | 1 | 1 |

# CountSketch

| $f_1$ | $f_2$ | $f_3$ | $f_4$ | $f_5$ | $f_6$ | $f_7$ |
|-------|-------|-------|-------|-------|-------|-------|
| 2     | 1     | 1     | 0     | 0     | 0     | 0     |

# 5

$h(x) = 3x + 2 \ (\mathrm{mod}\ 4)$
$s(x) = +1$ for $x \in \{1,2,3\}$
$s(x) = -1$ for $x \in \{4,5\}$

| $c_1$ | $c_2$ | $c_3$ | $c_4$ |
|-------|-------|-------|-------|
| 2     | 0     | 1     | 1     |

# CountSketch

| $f_1$ | $f_2$ | $f_3$ | $f_4$ | $f_5$ | $f_6$ | $f_7$ |
|-------|-------|-------|-------|-------|-------|-------|
| 2     | 1     | 1     | 0     | 1     | 0     | 0     |

5

$h(x) = 3x + 2 \pmod 4$

$s(x) = +1$ for $x \in \{1,2,3\}$

$s(x) = -1$ for $x \in \{4,5\}$

| $c_1$ | $c_2$ | $c_3$ | $c_4$ |
|-------|-------|-------|-------|
| 1     | 0     | 1     | 1     |

# CountSketch

| $f_1$ | $f_2$ | $f_3$ | $f_4$ | $f_5$ | $f_6$ | $f_7$ |
|-------|-------|-------|-------|-------|-------|-------|
| 2 | 1 | 1 | 0 | 1 | 0 | 0 |

- What is the estimation for $f_4$?
- What about $f_3$?
- What about $f_5$? What about $f_1$?

$h(x) = 3x + 2 \ (\text{mod } 4)$
$s(x) = +1$ for $x \in \{1,2,3\}$
$s(x) = -1$ for $x \in \{4,5\}$

| $c_1$ | $c_2$ | $c_3$ | $c_4$ |
|-------|-------|-------|-------|
| 1 | 0 | 1 | 1 |