CSCE 689: Special Topics in Modern Algorithms for Data Science

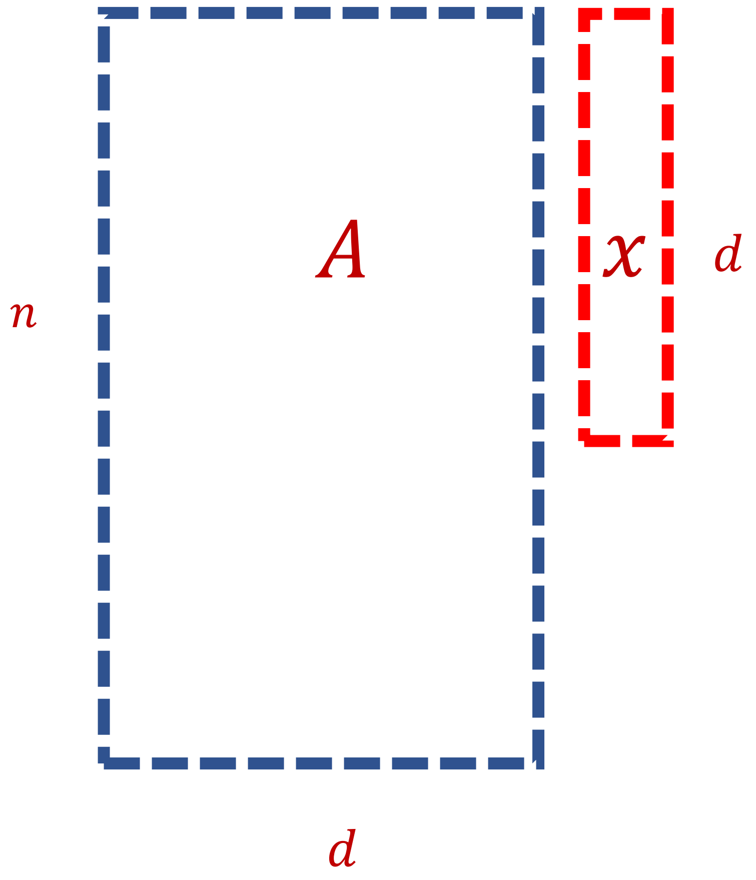
Lecture 30

Samson Zhou

Presentation Schedule

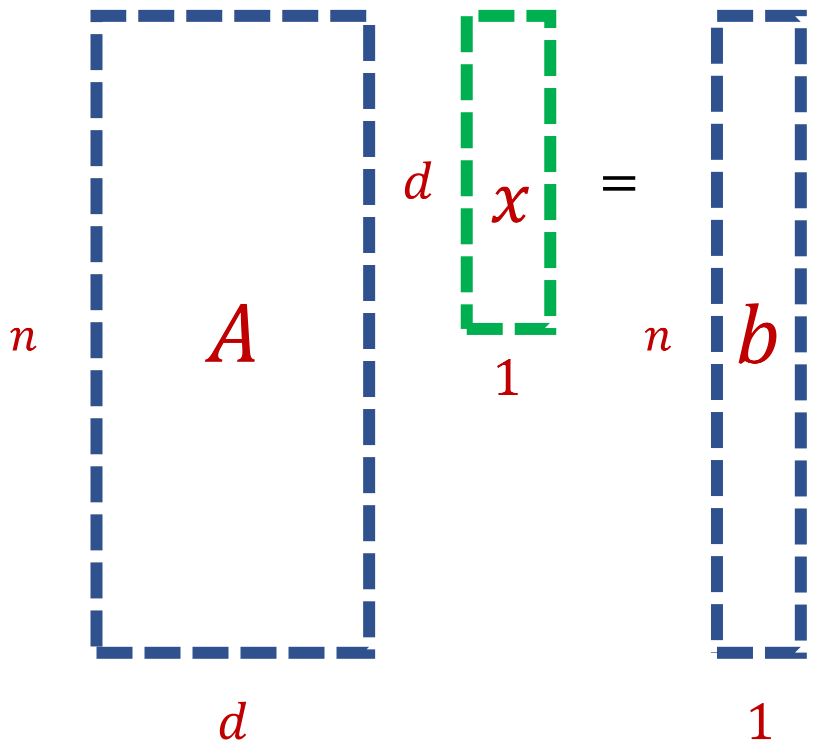
- November 27: Chunkai, Jung, Galaxy AI
- November 29: STMI, Anmol, Jason
- December 1: Bokun, Ayesha, Dawei, Lipai

Linear Algebra Review



- For $y = Ax$, we have $y_i = \langle a_i, x \rangle$

Regression



- Let A be a set of n observations, each with d features
- Let b be the vector of outcomes/labels for each observation

- Find the vector x such that $Ax = b$

Karl Rohe @karlrohe · 3h

i'm teaching linear regression in the spring

First thing we'll do: highlight the two *very* different motivations for linear models

1) for causal inference
2) for prediction

Breiman's "two cultures" paper is too advanced for mid-undergrads... are there any easier readings?

15 15 234 26K

Regression

- Economics and Finance:
  - Stock price prediction: Predict stock prices based on historical data and other relevant factors.
  - Econometric analysis: Study the relationships between economic variables like GDP, inflation, and interest rates.

Regression

- Medicine and Healthcare:
  - Medical research: Understand the relationship between factors such as age, genetics, and lifestyle on health outcomes.
  - Disease prediction: Predicting the probability of disease occurrence based on risk factors like smoking, diet, and exercise.

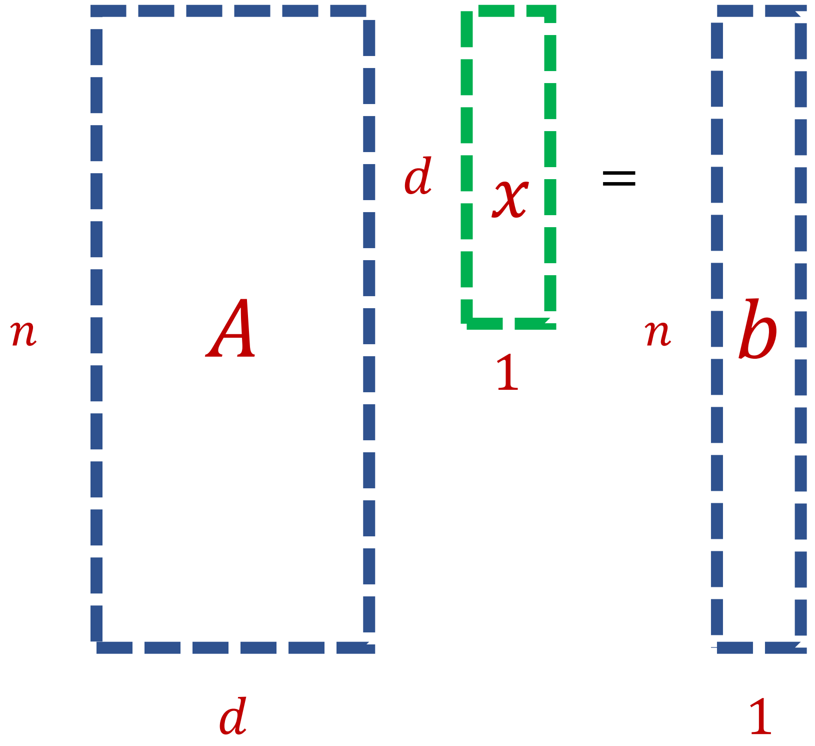
Regression

- Sports Analytics:
  - Player performance forecast: Predicting player performance in sports like baseball, basketball, or soccer based on historical data.
  - Game outcome prediction: Predicting the outcome of games based on team statistics and other factors.

Regression

- Environmental Science:
  - Climate modeling: Model climate change variables like temperature, precipitation, and sea levels.
  - Pollution analysis: Analyze the relationship between pollution levels and various factors like industrial activity and population.

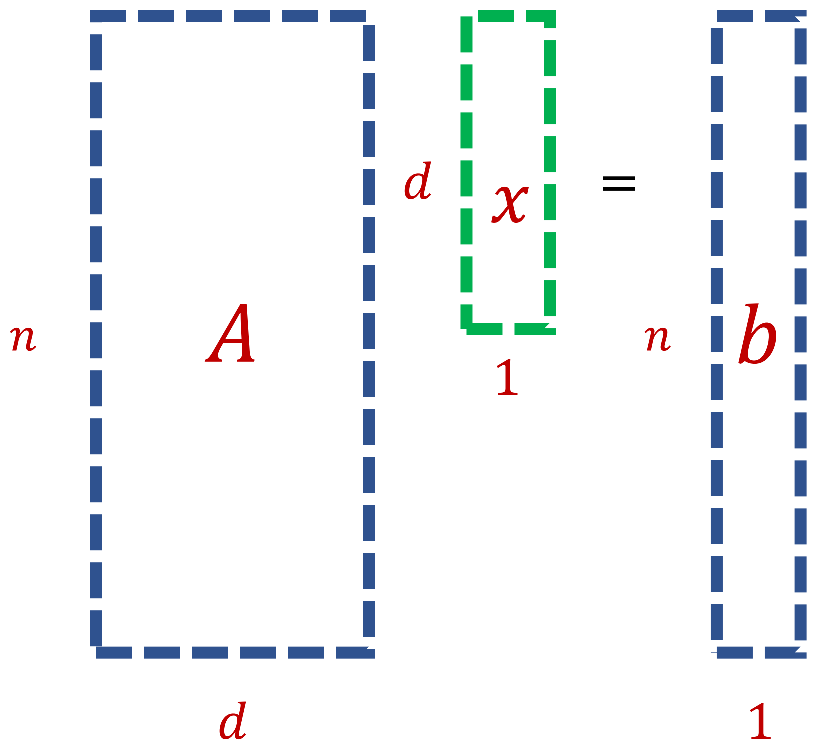
Regression



- Let A be a set of n observations, each with d features
- Let b be the vector of outcomes/labels for each observation

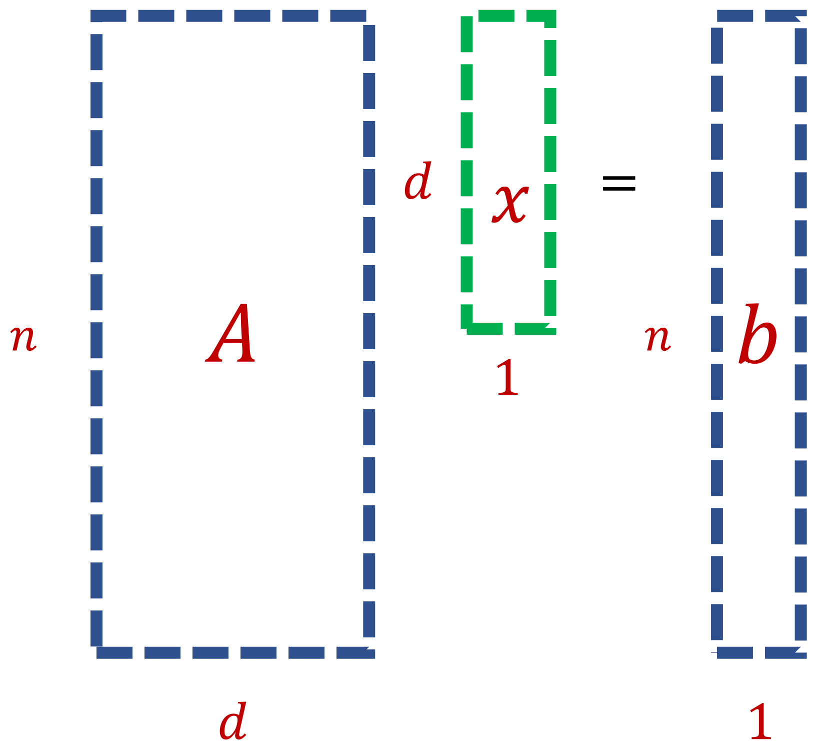
- Find the vector x such that $Ax = b$

Linear Algebra Review



- What are conditions for finding x such that $Ax = b$?

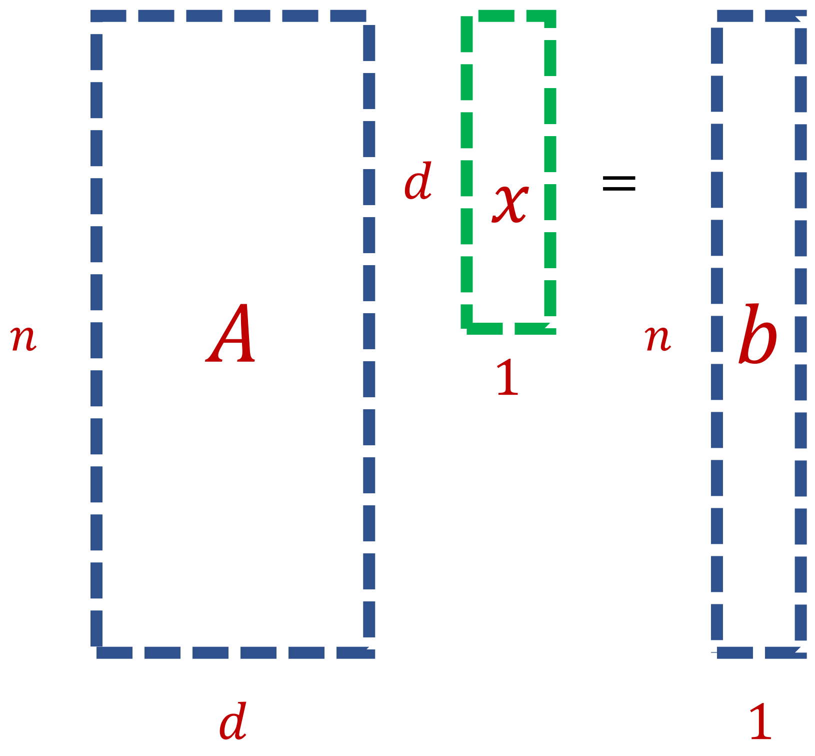
Linear Algebra Review



- What are conditions for finding x such that $Ax = b$?

- If the system is inconsistent, there are no solutions

Linear Algebra Review



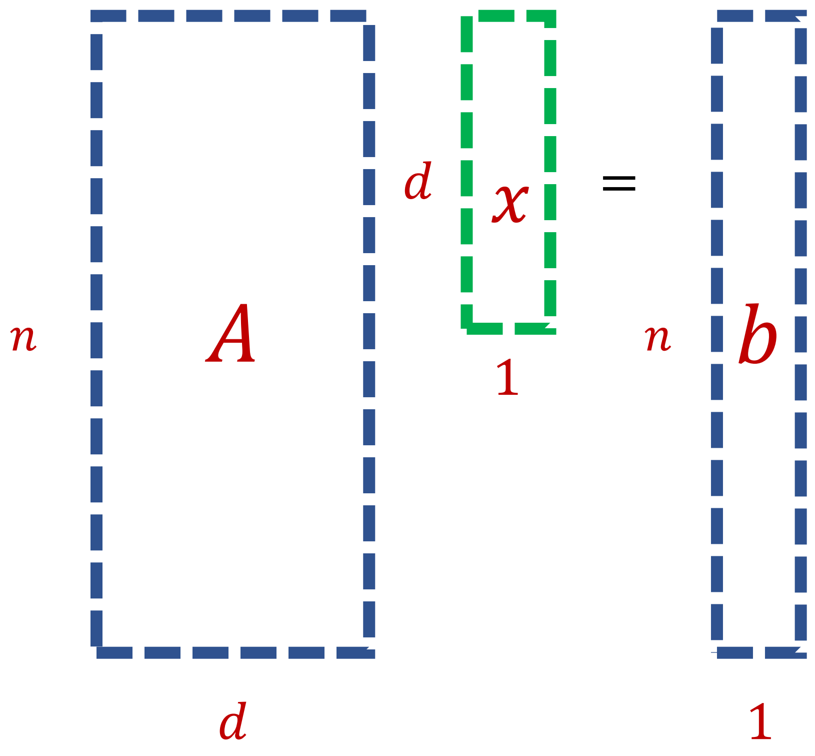
- What are conditions for finding x such that $Ax = b$?
- If the system is inconsistent, there are no solutions
- Can check for consistency by looking at the rank of the coefficient matrix and the augmented matrix

Linear Algebra Review

- For a square matrix $A \in \mathbb{R}^{n \times n}$, if A is full rank, i.e., $\text{rank}(A) = n$, then A has an inverse $A^{-1} \in \mathbb{R}^{n \times n}$ such that $AA^{-1} = A^{-1}A = \mathbb{I}_n$

- For a general matrix $A \in \mathbb{R}^{n \times d}$ with linearly independent columns, the Moore-Penrose inverse/pseudoinverse $A^{\dagger}$ of A satisfies $A^{\dagger} = (A^{\top}A)^{-1}A^{\top}$ so that $A^{\dagger}A = \mathbb{I}_d$

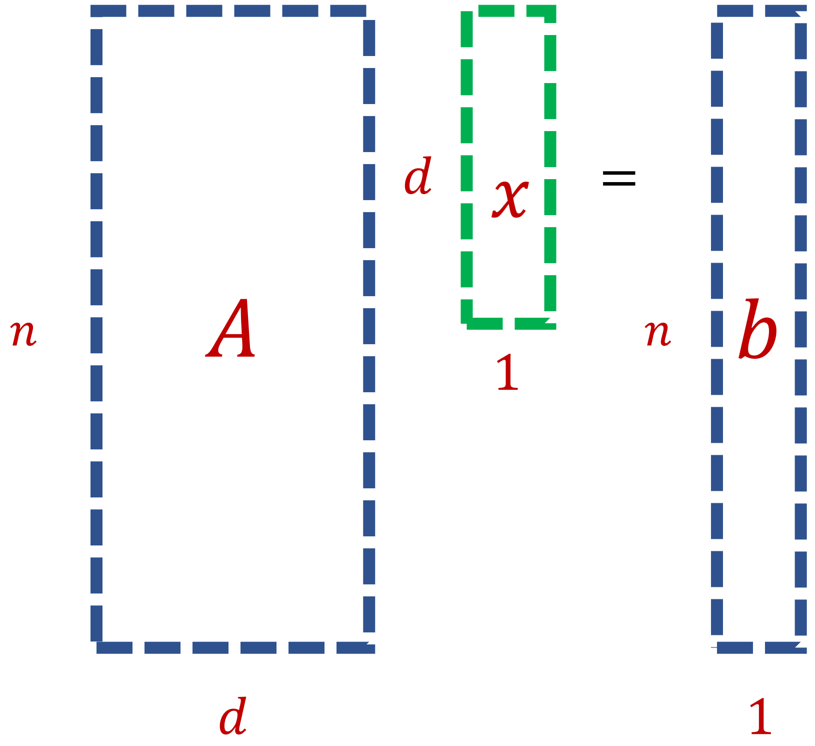
Linear Algebra Review



- What are conditions for finding x such that $Ax = b$?

- If the system is inconsistent, there are no solutions
- If the system is consistent and $n = d$, then there is a single solution, $x = A^{-1}b$

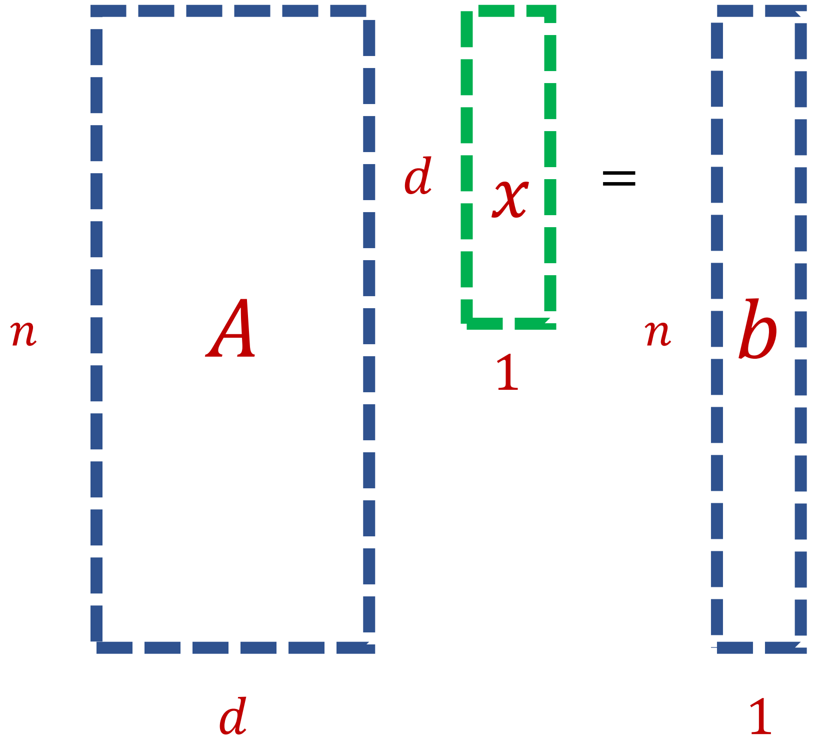
Linear Algebra Review



- What are conditions for finding x such that $Ax = b$?

- If the system is inconsistent, there are no solutions
- If the system is consistent and $n < d$, then are infinite solutions and can find a solution by looking at n columns of A

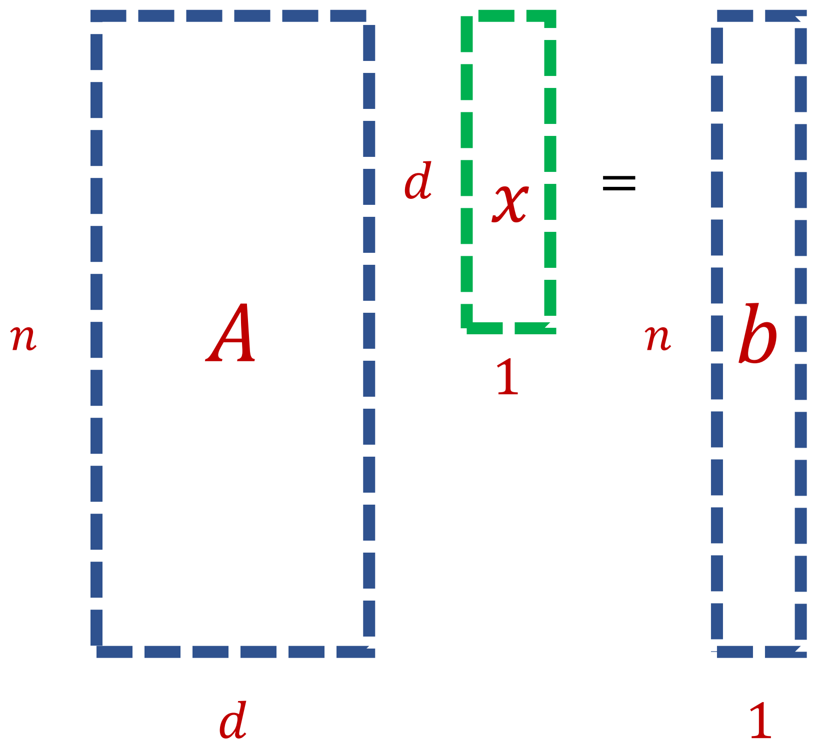
Linear Algebra Review



- What are conditions for finding x such that $Ax = b$?

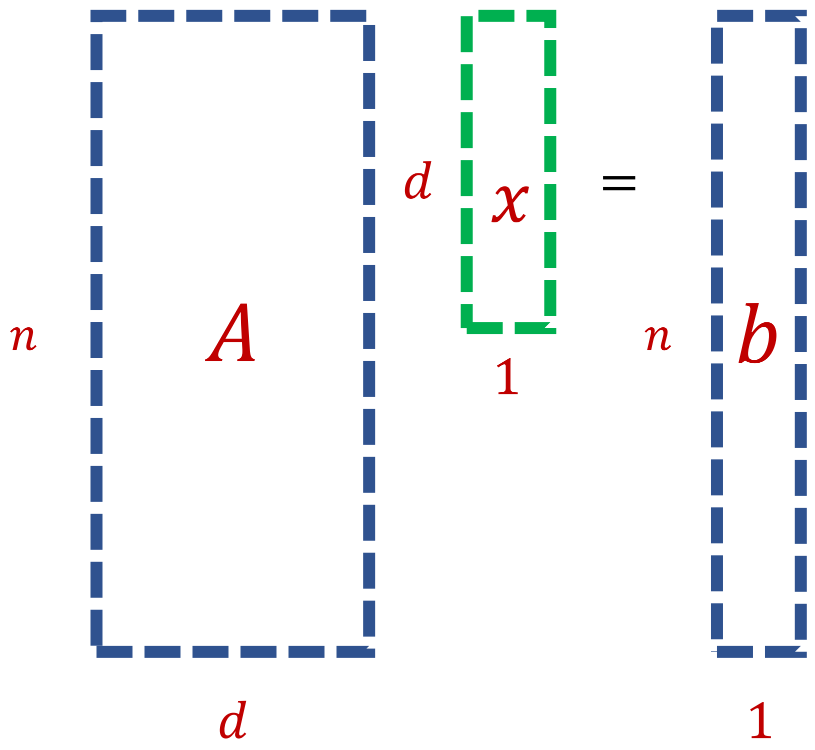
- If the system is inconsistent, there are no solutions
- If the system is consistent and $n > d$, then at most one solution ($x = A^{\dagger}b$) and can find by looking at d rows of A

Regression



- What are conditions for finding x such that $Ax = b$?
- What to do when there is no x such that $Ax = b$?

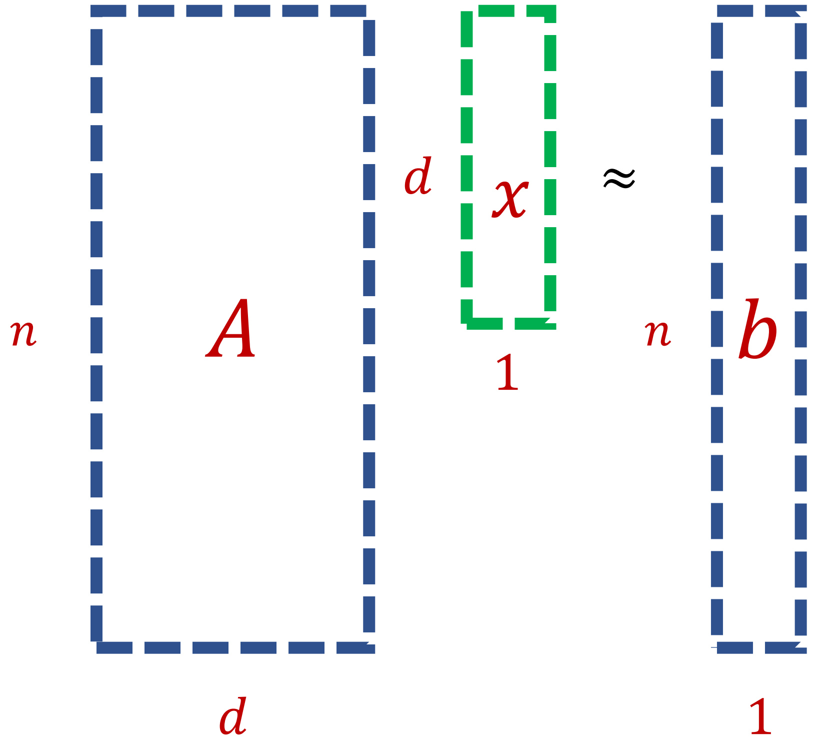
Regression



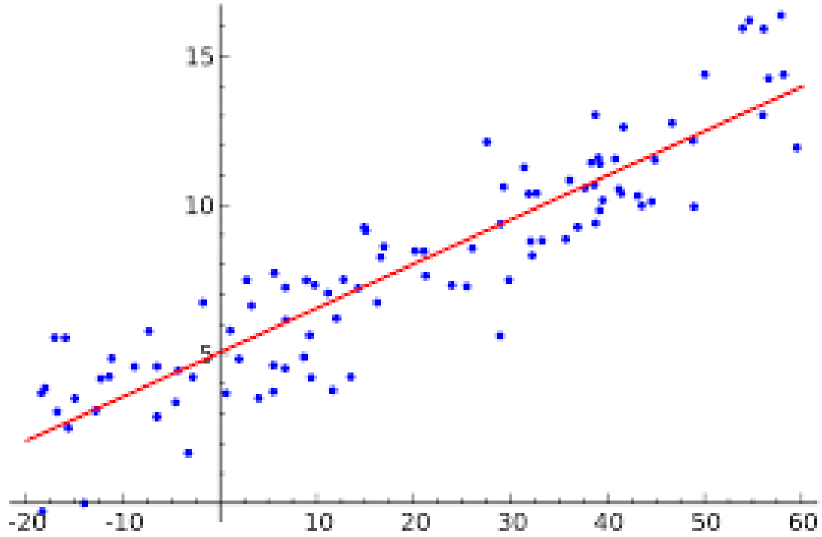
- What are conditions for finding x such that $Ax = b$?

- What to do when there is no x such that $Ax = b$?
- Minimize $\mathcal{L}(Ax - b)$ for some loss function $\mathcal{L}$

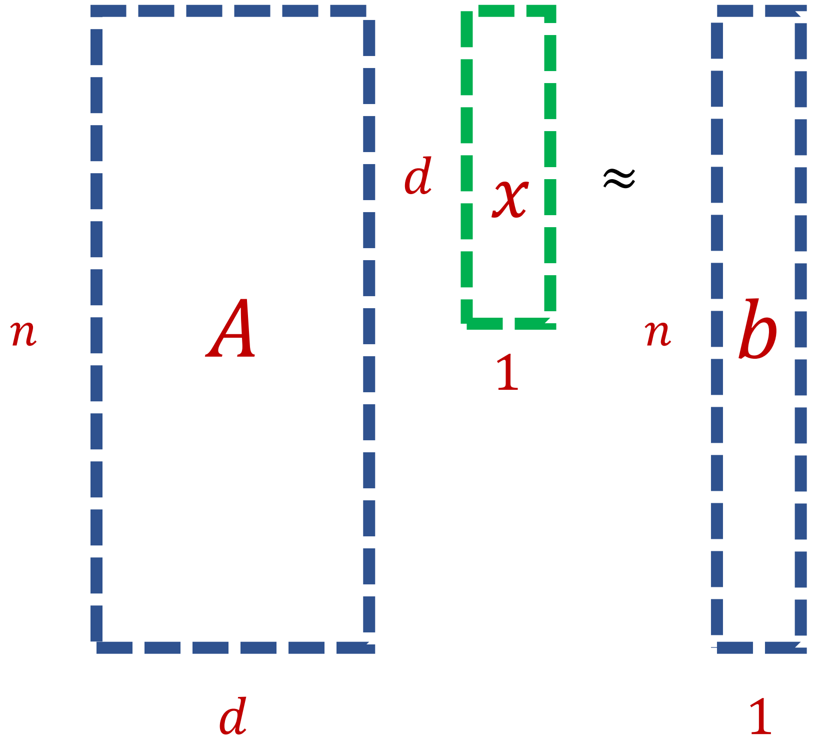
Linear Regression



- Find the vector x that minimizes $\|Ax - b\|_2$
- "Least squares" optimization



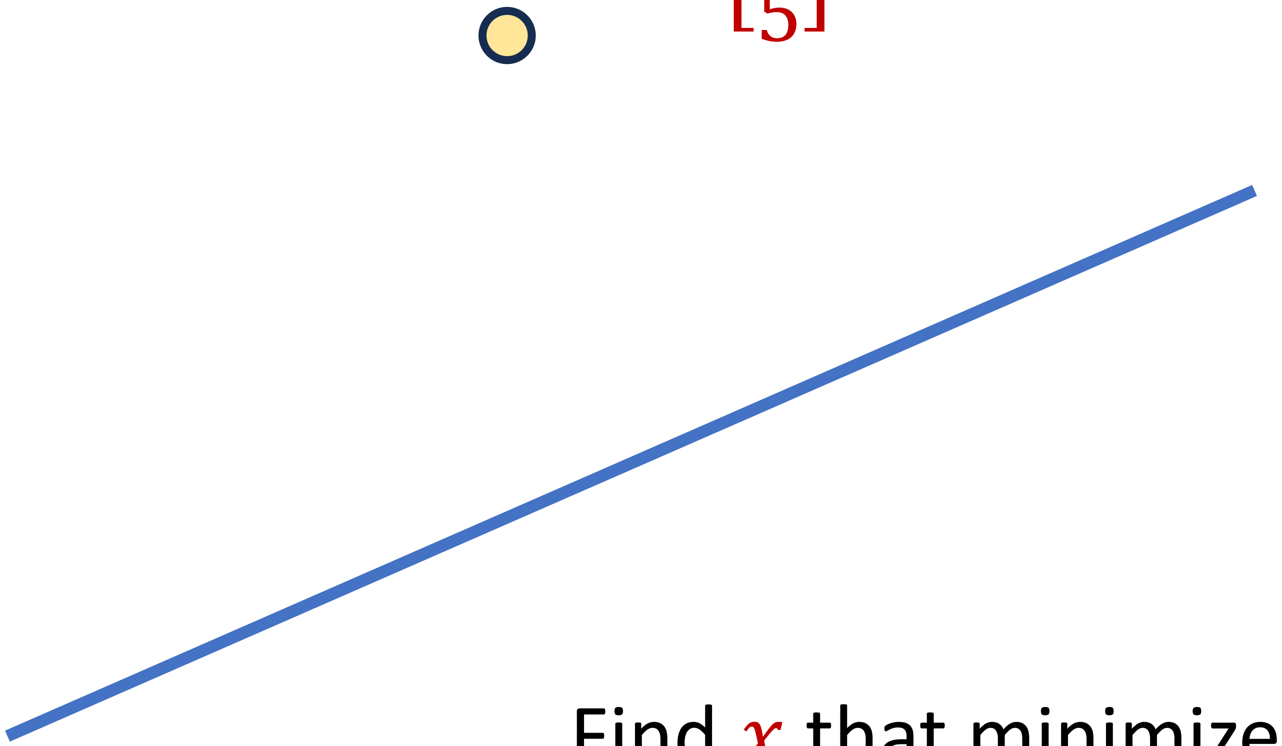
Linear Regression



- Find the vector x that minimizes $\|Ax - b\|_2$
- What is the solution to $\min_{x \in \mathbb{R}^d} \|Ax - b\|_2$?

Linear Regression

$b = \begin{bmatrix} 5 \\ 5 \end{bmatrix}$

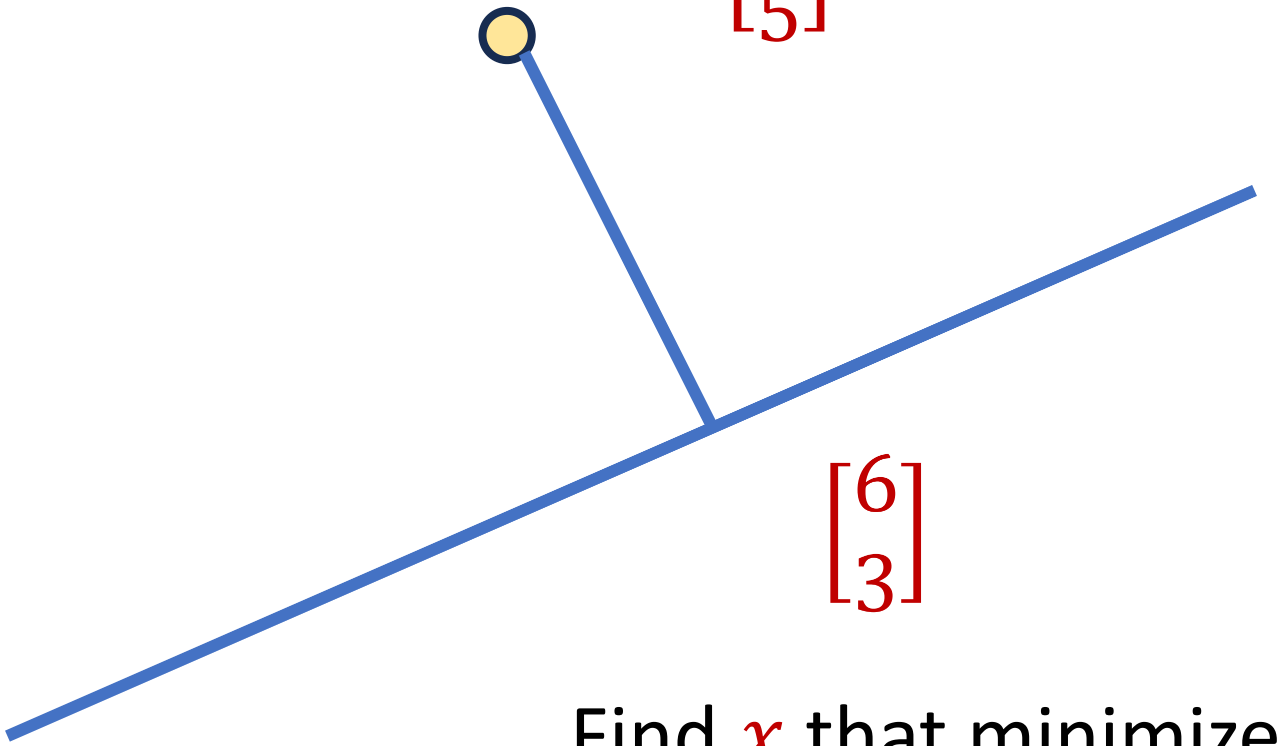


$A = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

Find x that minimizes $\|Ax - b\|_2$

Linear Regression

$b = \begin{bmatrix} 5 \\ 5 \end{bmatrix}$

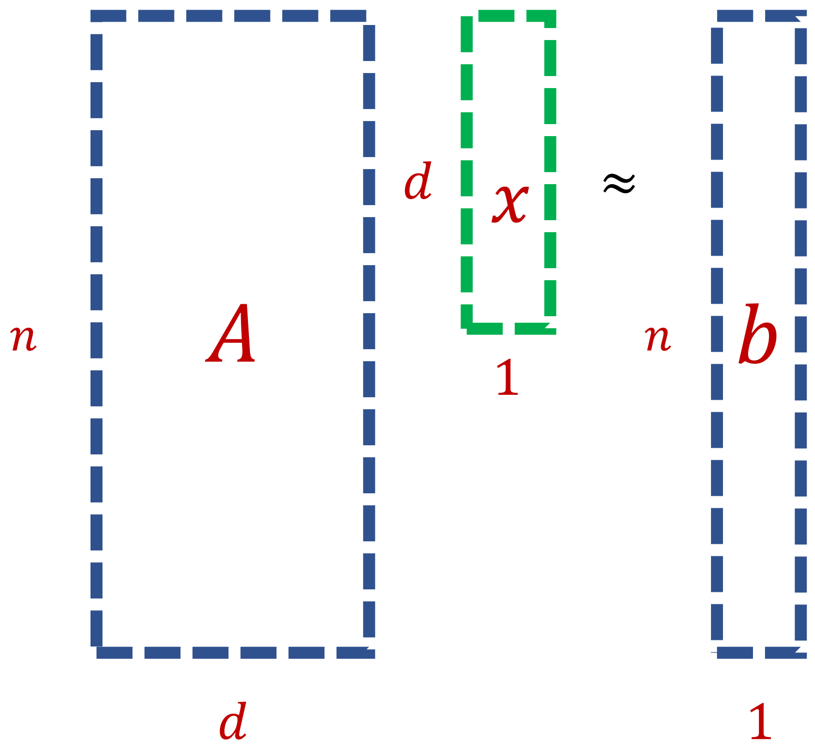


$\begin{bmatrix} 6 \\ 3 \end{bmatrix}$

$A = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

Find x that minimizes $\|Ax - b\|_2$

Linear Regression



- Find the vector x that minimizes $\|Ax - b\|_2$
- What is the solution to $\min_{x \in \mathbb{R}^d} \|Ax - b\|_2$?

- Let $b^{\perp}$ be the projection of b onto the column space of A
- Decompose b into $b^{\perp}$ and $b^{\parallel}$

Linear Regression

- We have $\arg\min_{x \in \mathbb{R}^d} \|Ax - b\|_2 = \arg\min_{x \in \mathbb{R}^d} \|Ax - b\|_2^2$

- $\|Ax - b\|_2^2 = \left\|Ax - b^{\perp} - b^{\parallel}\right\|_2^2$

Linear Regression

- We have $\arg\min_{x \in \mathbb{R}^d} \|Ax - b\|_2 = \arg\min_{x \in \mathbb{R}^d} \|Ax - b\|_2^2$

- $$\|Ax - b\|_2^2 = \left\|Ax - b^{\perp} - b^{\parallel}\right\|_2^2$$
$$= \left\|Ax - b^{\parallel}\right\|_2^2 - 2\left\langle Ax - b^{\parallel}, b^{\perp}\right\rangle + \|b^{\perp}\|_2^2$$

Linear Regression

- We have $\arg\min_{x\in\mathbb{R}^d} \|Ax-b\|_2 = \arg\min_{x\in\mathbb{R}^d} \|Ax-b\|_2^2$

- $\|Ax-b\|_2^2 = \left\|Ax-b^{\perp}-b^{\parallel}\right\|_2^2$

$$= \left\|Ax-b^{\parallel}\right\|_2^2 - 2\left\langle Ax-b^{\parallel}, b^{\perp}\right\rangle + \|b^{\perp}\|_2^2$$

$$= \left\|Ax-b^{\parallel}\right\|_2^2 + \|b^{\perp}\|_2^2$$

Linear Regression

- We have $\arg\min_{x \in \mathbb{R}^d} \|Ax - b\|_2 = \arg\min_{x \in \mathbb{R}^d} \|Ax - b\|_2^2$

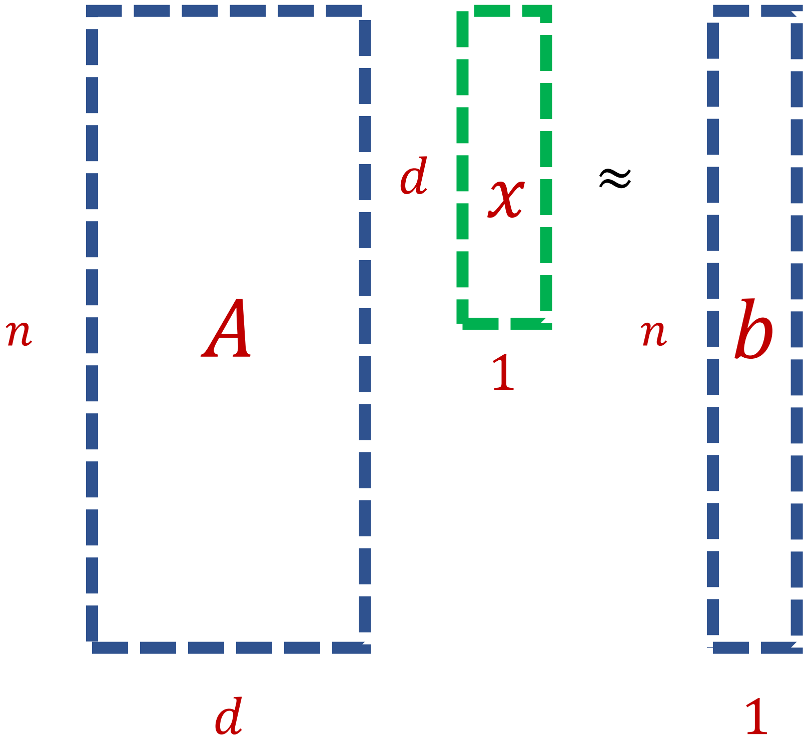
- $\|Ax - b\|_2^2 = \left\|Ax - b^{\perp} - b^{\parallel}\right\|_2^2$

$$= \left\|Ax - b^{\parallel}\right\|_2^2 - 2\langle Ax - b^{\parallel}, b^{\perp}\rangle + \|b^{\perp}\|_2^2$$

$$= \left\|Ax - b^{\parallel}\right\|_2^2 + \|b^{\perp}\|_2^2$$

- Minimized for $\left\|Ax - b^{\parallel}\right\|_2^2 = 0$ when $x = A^{\dagger} b^{\parallel} = A^{\dagger} b$

Linear Regression



- Find the vector x that minimizes $\|Ax - b\|_2$
- "Least squares" optimization
- MLE under Gaussian noise
- Closed form solution: $x = A^{\dagger} b$