Tight Bounds for Adversarially Robust Streams and Sliding Windows via Difference Estimators

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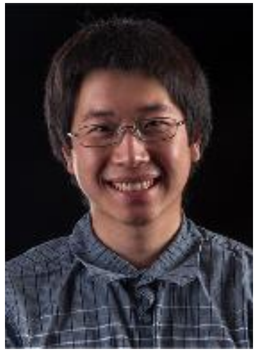

Carnegie Mellon University

Model #1: Streaming Model

- Input: Elements of an underlying data set S, which arrives sequentially
- Output: Evaluation (or approximation) of a given function
- Goal: Use space *sublinear* in the size m of the input S

1 0 1 1 1 0 0 1

Heavy-Hitters

- Given a set S of m elements from $[n]$, let f_i be the frequency of element i. (How often it appears)

$$1\ 1\ 2\ 1\ 2\ 1\ 1\ 2\ 3 \rightarrow [5, 3, 1, 0] \coloneqq f$$

Heavy-Hitters

- Given a set S of m elements from $[n]$, let f_i be the frequency of element i. (How often it appears)
- Let L_2 be the norm of the frequency vector:

$$L_2 = \sqrt{f_1^2 + f_2^2 + \cdots + f_n^2}$$

- Goal: Given a set S of m elements from $[n]$ and a threshold ϵ, output the elements i such that $f_i > \epsilon\, L_2$...and no elements j such that $f_j < \frac{\epsilon}{16}\, L_2$
- Motivation: DDoS prevention, iceberg queries

Frequency Moments

- Given a set S of m elements from $[n]$, let f_i be the frequency of element i. (How often it appears)
- Let F_p be the frequency moment of the vector:

$$F_p = f_1^p + f_2^p + \cdots + f_n^p$$

- Goal: Given a set S of m elements from $[n]$ and an accuracy parameter ϵ, output a $(1+\epsilon)$-approximation to F_p
- Motivation: Entropy estimation, linear regression

Distinct Elements

- Given a set S of m elements from $[n]$, let f_i be the frequency of element i. (How often it appears)
- Let F_0 be the frequency moment of the vector:

$$F_0 = |\{i : f_i \neq 0\}|$$

- Goal: Given a set S of m elements from $[n]$ and an accuracy parameter ϵ, output a $(1+\epsilon)$-approximation to F_0
- Motivation: Traffic monitoring

$(1+\epsilon)$-Approximation Streaming Algorithms

- Space $O\left(\frac{1}{\epsilon^2} + \log n\right)$ algorithm for F_0 [KaneNelsonWoodruff10], [Blasiok20]
- Space $O\left(\frac{1}{\epsilon^2} \log n\right)$ algorithm for F_p with $p \in (0, 2]$ [BlasiokDingNelson17]
- Space $O\left(\frac{1}{\epsilon^2} n^{1-2/p} \log^2 n\right)$ algorithm for F_p with $p > 2$ [Ganguly11, GangulyWoodruff18]
- Space $O\left(\frac{1}{\epsilon^2} \log n\right)$ algorithm for L_2-heavy hitters [BravermanChestnutIvkinNelsonWangWoodruff17]

Model #2: Adversarially Robust Streaming

- **Input**: Elements of an underlying data set S, which arrives sequentially and *adversarially*
- **Output**: Evaluation (or approximation) of a given function
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- Adversarially Robust: "Future queries may depend on previous queries"
- Motivation: Database queries, adversarial ML

$(1+\epsilon)$-Robust Algorithms [Ben-EliezerJayaramWoodruffYogev20]

- Space $\tilde{O}\left(\frac{1}{\epsilon^3}\log n\right)$ algorithm for F_0
- Space $\tilde{O}\left(\frac{1}{\epsilon^3}\log n\right)$ algorithm for F_p with $p \in (0,2]$
- Space $\tilde{O}\left(\frac{1}{\epsilon^3}n^{1-2/p}\right)$ algorithm for F_p with $p > 2$
- Space $\tilde{O}\left(\frac{1}{\epsilon^3}\log n\right)$ algorithm for L_2-heavy hitters

"A general framework that loses* nothing in n and only $\frac{1}{\epsilon}$"

“What’s an epsilon between friends?

- Statista: $\sim 300B$ e-mails sent per day
- Unsigned integer range: $n = 2^{32} \sim 4B$
- Accuracy: $\epsilon = 0.01$
- Since $\frac{1}{\epsilon} > \log n$, we should care about $\frac{1}{\epsilon}$ factors!

$(1+\epsilon)$-Robust Algorithms [HassidimKaplanMansourMatiasStemmer20]

- Space $\tilde{O}\left(\frac{1}{\epsilon^{2.5}}\log^4 n\right)$ algorithm for F_0
- Space $\tilde{O}\left(\frac{1}{\epsilon^{2.5}}\log^4 n\right)$ algorithm for F_p with $p \in (0,2]$
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"$\frac{1}{\epsilon}$ losses are not necessary"

Our Results: $(1+\epsilon)$-Robust Algorithms

- Space $\tilde{O}\left(\frac{1}{\epsilon^2}\log n\right)$ algorithm for F_0
- Space $\tilde{O}\left(\frac{1}{\epsilon^2}\log n\right)$ algorithm for F_p with $p \in (0,2]$
- Space $\tilde{O}\left(\frac{1}{\epsilon^2}n^{1-2/p}\right)$ algorithm for F_p with integer $p > 2$
- Space $\tilde{O}\left(\frac{1}{\epsilon^2}\log n\right)$ algorithm for L_2-heavy hitters

"No losses* are necessary!"

Summary: $(1+\epsilon)$-Robust Algorithms

Problem	[BJWY20] Space	[HKM+20] Space	Our Result
Distinct Elements	$\tilde{\mathcal{O}}\left(\frac{\log n}{\varepsilon^3}\right)$	$\tilde{\mathcal{O}}\left(\frac{\log^4 n}{\varepsilon^{2.5}}\right)$	$\tilde{\mathcal{O}}\left(\frac{\log n}{\varepsilon^2}\right)$
F_p Estimation, $p \in (0,2]$	$\tilde{\mathcal{O}}\left(\frac{\log n}{\varepsilon^3}\right)$	$\tilde{\mathcal{O}}\left(\frac{\log^4 n}{\varepsilon^{2.5}}\right)$	$\tilde{\mathcal{O}}\left(\frac{\log n}{\varepsilon^2}\right)$
Shannon Entropy	$\tilde{\mathcal{O}}\left(\frac{\log^6 n}{\varepsilon^5}\right)$	$\tilde{\mathcal{O}}\left(\frac{\log^4 n}{\varepsilon^{3.5}}\right)$	$\tilde{\mathcal{O}}\left(\frac{\log^3 n}{\varepsilon^2}\right)$
L_2-Heavy Hitters	$\tilde{\mathcal{O}}\left(\frac{\log n}{\varepsilon^3}\right)$	$\tilde{\mathcal{O}}\left(\frac{\log^4 n}{\varepsilon^{2.5}}\right)$	$\tilde{\mathcal{O}}\left(\frac{\log n}{\varepsilon^2}\right)$
F_p Estimation, integer $p > 2$	$\tilde{\mathcal{O}}\left(\frac{n^{1-2/p}}{\varepsilon^3}\right)$	$\tilde{\mathcal{O}}\left(\frac{n^{1-2/p}}{\varepsilon^{2.5}}\right)$	$\tilde{\mathcal{O}}\left(\frac{n^{1-2/p}}{\varepsilon^2}\right)$
F_p Estimation, $p \in (0,2]$, flip number λ	$\tilde{\mathcal{O}}\left(\frac{\lambda \log^2 n}{\varepsilon^2}\right)$	$\tilde{\mathcal{O}}\left(\frac{\log^3 n\sqrt{\lambda \log n}}{\varepsilon^2}\right)$	$\tilde{\mathcal{O}}\left(\frac{\lambda \log^2 n}{\varepsilon}\right)$

Model #3: Sliding Window Model

- Input: Elements of an underlying data set S, which arrives sequentially
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- Sliding Window: “Only the m most recent updates form the underlying data set S”

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- **Output**: Evaluation (or approximation) of a given function
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- **Sliding Window**: "Only the m most recent updates form the underlying data set S"
  - Emphasizes recent interactions, appropriate for time sensitive settings

1 0 1 [1 1 0 0 1 1 0 1]

$(1+\epsilon)$-Approximation Sliding Window Algorithms

- Space $\tilde{O}\left(\frac{1}{\epsilon^2}\log n\right)$ algorithm for F_0 [BravermanGrigorescuLangWoodruffZhou18]
- Space $\tilde{O}\left(\frac{1}{\epsilon^2}\log^3 n\right)$ algorithm for L_2-heavy hitters [BravermanGrigorescuLangWoodruffZhou18]

$(1+\epsilon)$-Approximation Sliding Window Algorithms

- Space $O\left(\frac{1}{\epsilon^3}\log^3 n\right)$ algorithm for F_p with $p \in (0,1)$ [BravermanOstrovsky07]
- Space $O\left(\frac{1}{\epsilon^{2+p}}\log^3 n\right)$ algorithm for F_p with $p \in (1,2]$ [BravermanOstrovsky07]
- Space $\tilde{O}\left(\frac{1}{\epsilon^{2+p}} n^{1-2/p}\right)$ algorithm for F_p with $p > 2$ [BravermanOstrovsky07]

"A general framework that loses* nothing in n and only $\frac{1}{\epsilon}$"

Our Results: $(1+\epsilon)$-Approximation Sliding Window Algorithms

❖ Space $\tilde{O}\left(\frac{1}{\epsilon^2}\log^3 n\right)$ algorithm for F_p with $p \in (0,2]$

Problem	[BO07] Space	Our Result
L_p Estimation, $p \in (0,1)$	$\tilde{\mathcal{O}}\left(\frac{\log^3 n}{\varepsilon^3}\right)$	$\tilde{\mathcal{O}}\left(\frac{\log^3 n}{\varepsilon^2}\right)$
L_p Estimation, $p \in (1,2]$	$\tilde{\mathcal{O}}\left(\frac{\log^3 n}{\varepsilon^{2+p}}\right)$	$\tilde{\mathcal{O}}\left(\frac{\log^3 n}{\varepsilon^2}\right)$
L_p Estimation, integer $p > 2$	$\tilde{\mathcal{O}}\left(\frac{n^{1-2/p}}{\varepsilon^{2+p}}\right)$	$\tilde{\mathcal{O}}\left(\frac{n^{1-2/p}}{\varepsilon^2}\right)$
Entropy Estimation	$\tilde{\mathcal{O}}\left(\frac{\log^5 n}{\varepsilon^4}\right)$	$\tilde{\mathcal{O}}\left(\frac{\log^5 n}{\varepsilon^2}\right)$

"$\frac{1}{\epsilon}$ losses are not necessary"

Format

- Part 1: Background
- Part 2: Frameworks
- Part 3: Difference Estimators

Questions?

AMS F_2 Algorithm

- Let $s \in \{-1, +1\}^n$ be a sign vector of length n
- Let $Z = \langle s, f \rangle = s_1 f_1 + \cdots + s_n f_n$ and consider Z^2

$$E[Z^2] = \sum_{i,j} E\left[s_i s_j f_i f_j\right] = f_1^2 + \cdots + f_n^2$$

$$Var[Z^2] \leq \sum_{i,j} E\left[s_i s_j s_k s_l f_i f_j f_k f_l\right] \leq 2F_2^2$$

- Take the mean of $O\left(\frac{1}{\epsilon^2}\right)$ inner products for $(1 + \epsilon)$-approximation [AlonMatiasSzegedy99]

“Attack” on AMS

- Can learn whether $s_i = s_j$ from $\langle s, e_i + e_j \rangle$
- Let $f_i = 1$ if $s_i = s_1$ and $f_i = -1$ if $s_i \neq s_1$
- $Z = \langle s, f \rangle = s_1 f_1 + \cdots + s_n f_n = m$ and $Z^2 = m^2$ deterministically

- What happened? Randomness of algorithm not independent of input

Reconstruction Attack on Linear Sketches

- Linear sketches are "not robust" to adversarial attacks, must use $\Omega(n)$ space [HardtWoodruff13]
- Approximately learn sketch matrix U, query something in the kernel of U
- Iterative process, start with $V_1, \ldots, V_r$
- Correlation finding: Find vectors weakly correlated with U orthogonal to V_{i-1}
- Boosting: Use these vectors to find strongly correlated vector v
- Progress: Set $V_i = \text{span}(V_{i-1}, v)$

Insertion-Only Streams

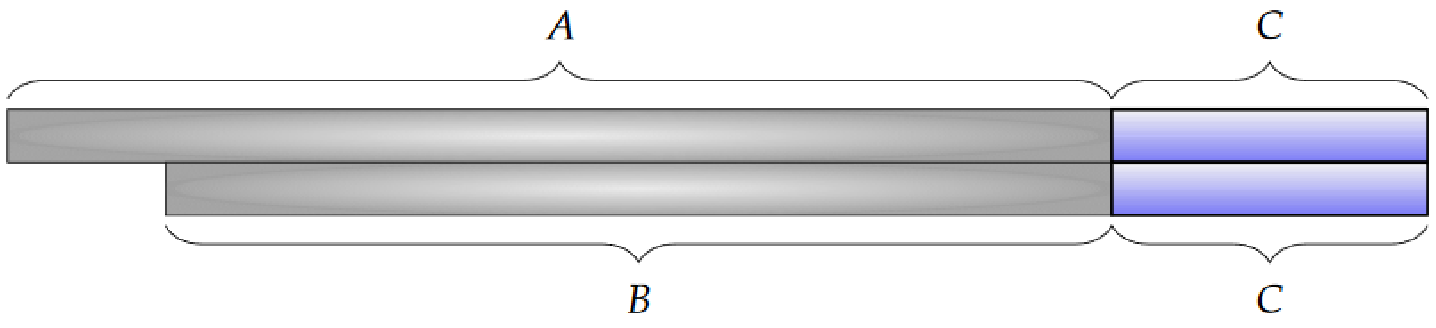
- Key: Deletions are needed to perform this attack
- Similar lower bounds for the sliding window model [DatarGionisIndykMotwani02]

- Assume insertion-only updates
- How do the previous results work?

Sliding Window Algorithms

❖ Suppose we are trying to approximate some given function

1. Suppose we have a streaming algorithm for this function
2. Suppose this function is "smooth": If $f(B)$ is a "good" approximation to $f(A)$, then $f(B \cup C)$ will always be a "good" approximation to $f(A \cup C)$.



❖ Smooth histogram framework [BravermanOstrovsky07] gives a sliding window algorithm for this function

Smooth Histogram

- Suppose we are trying to approximate some given function
- Smooth histogram framework [BO07] gives a sliding window algorithm for this function
- Start a new instance of the streaming algorithm (along with existing instances) each time a new element arrives
- Each time there are three instances that report “close” values, delete the middle one
- Use different checkpoints to “sandwich” the sliding window

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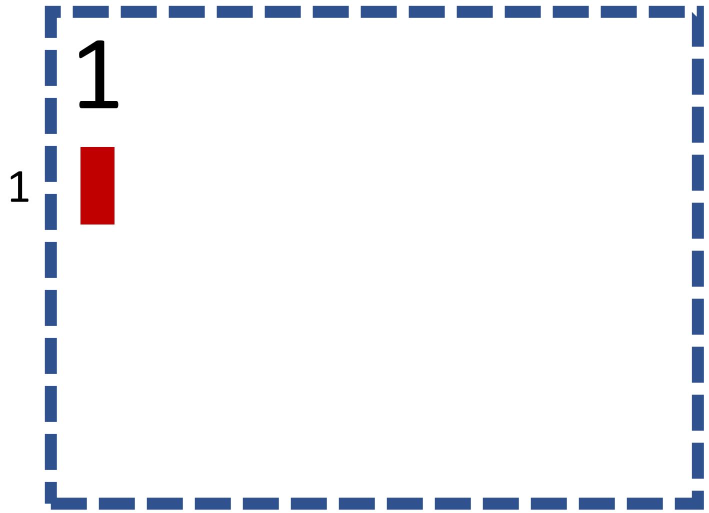

- Example: Number of ones in sliding window (2-approximation)

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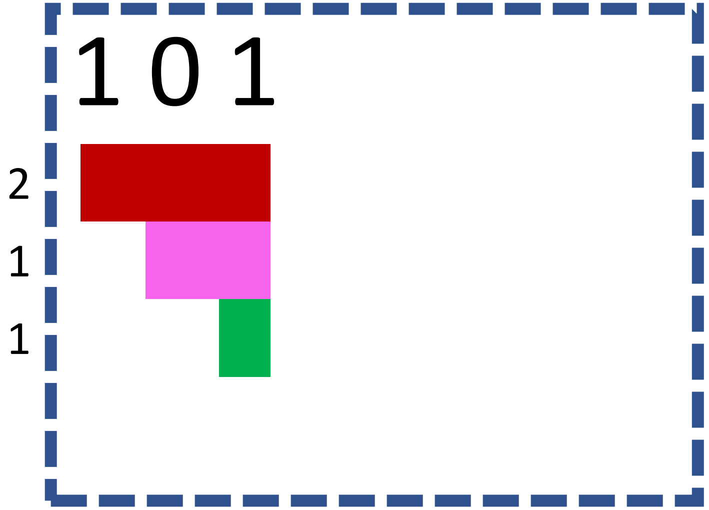

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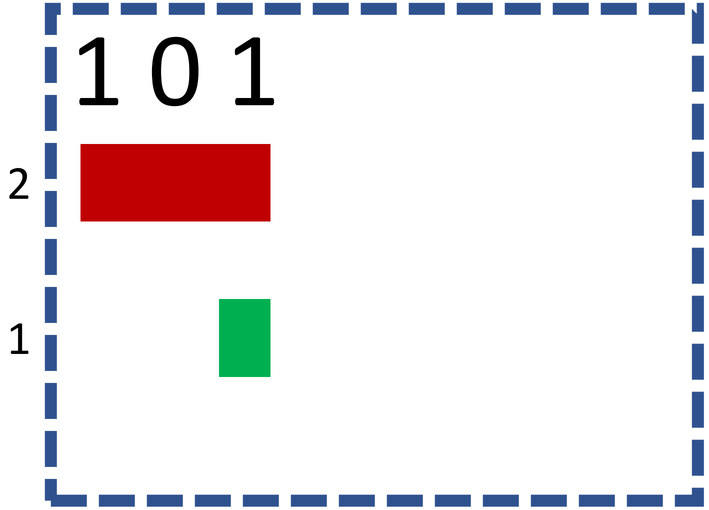

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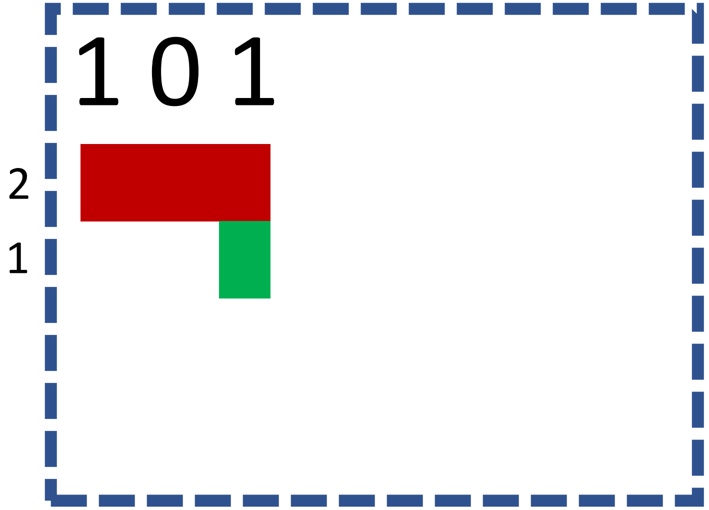

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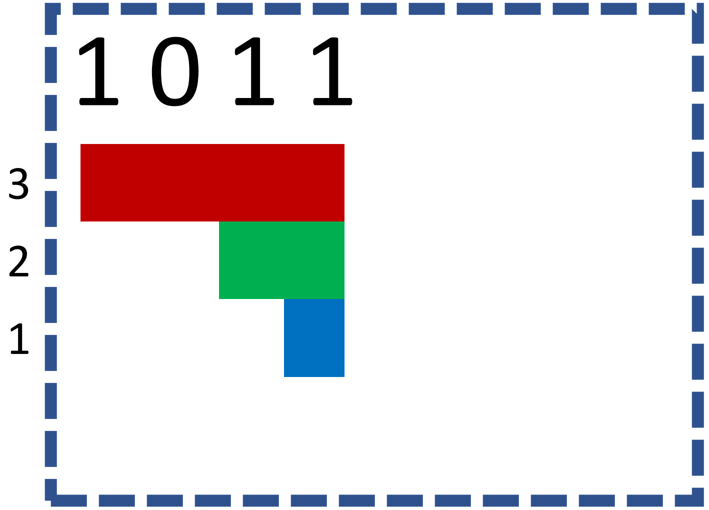

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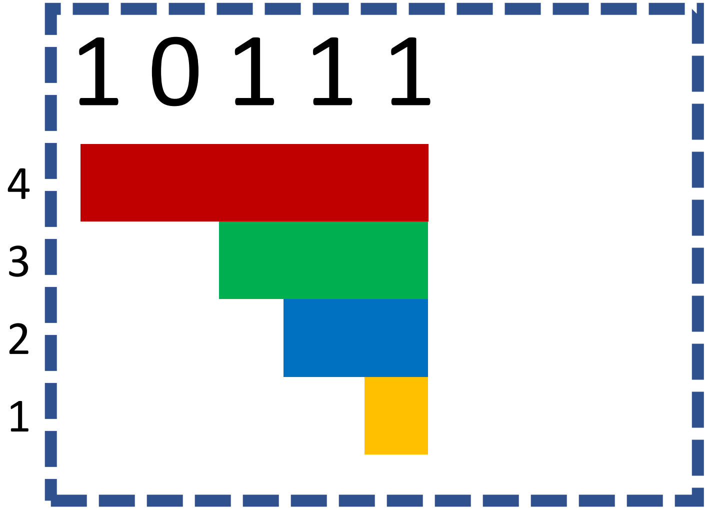

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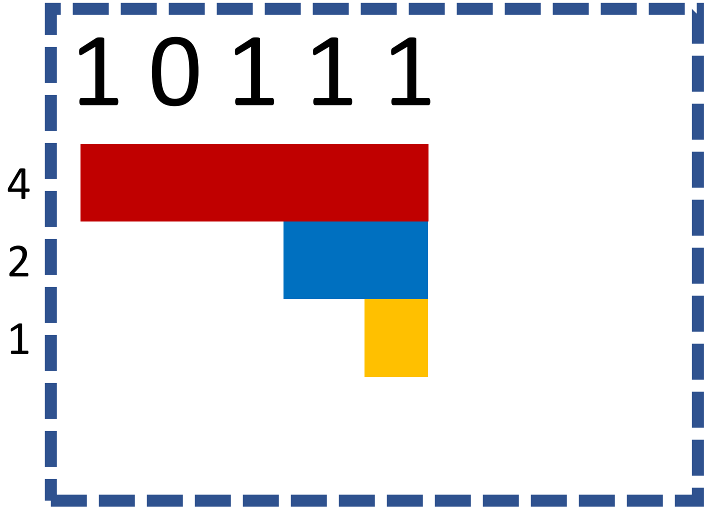

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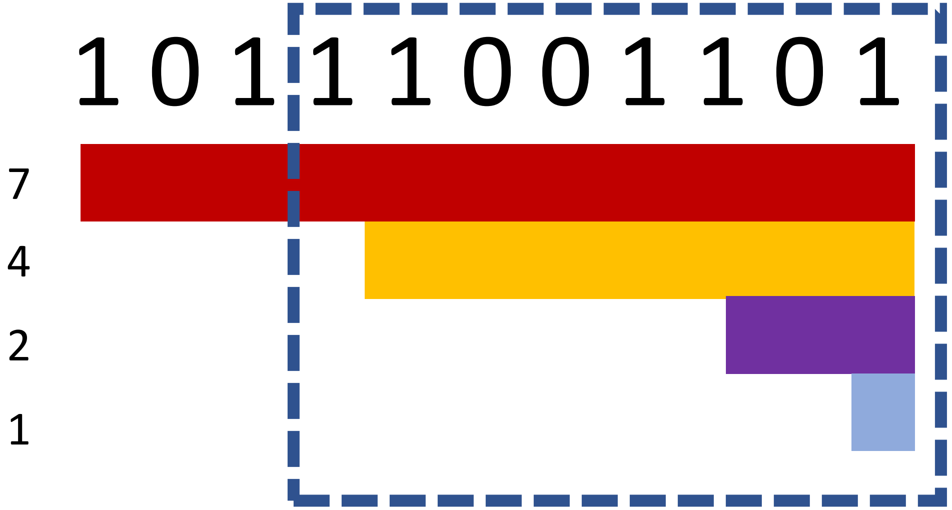

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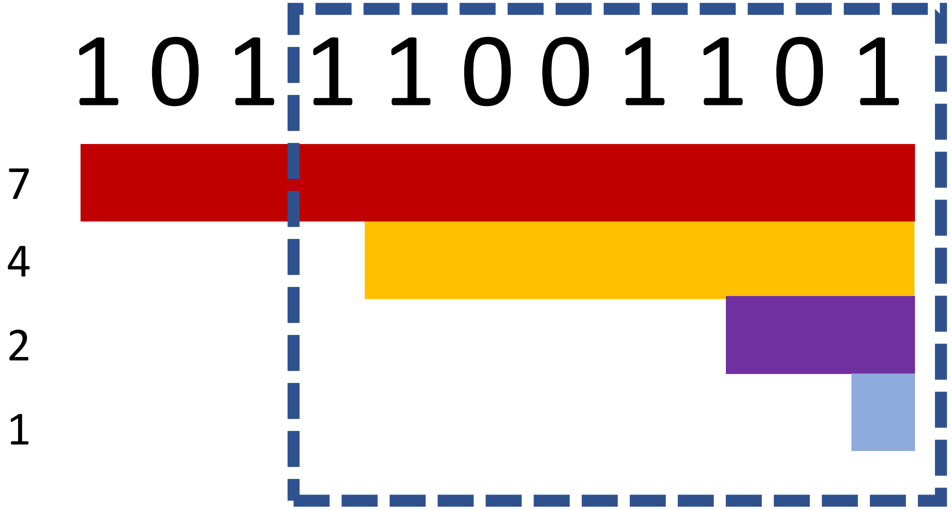

- ❖ Example: Number of ones in sliding window (2-approximation)
- ❖ Number of ones in sliding window is at least 4 and at most 7
- ❖ 4 is a good approximation

Robust Algorithms

- Suppose we are trying to approximate some given function
  1. Suppose we have a streaming algorithm for this function
  2. Suppose this function is monotonic and the stream is insertion-only
- Sketch switching framework [Ben-EliezerJayaramWoodruffYogev20] gives a robust for this function
- Start many instances of the streaming algorithm at the beginning
- Use an instance of the algorithm but "freeze" the output
- Each time the next instance has value $(1 + O(\epsilon))$ more than the "frozen" output, use the next instance and "freeze" its output

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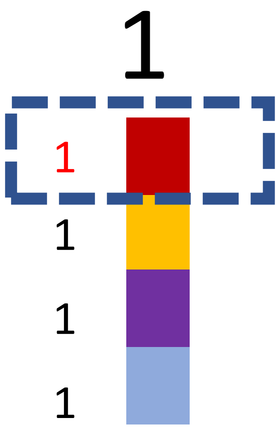

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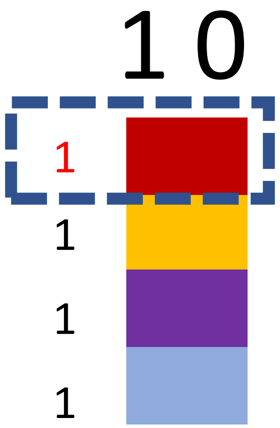

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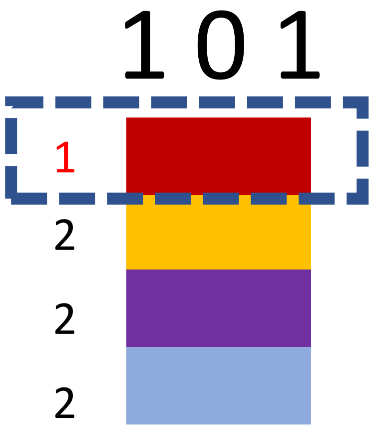

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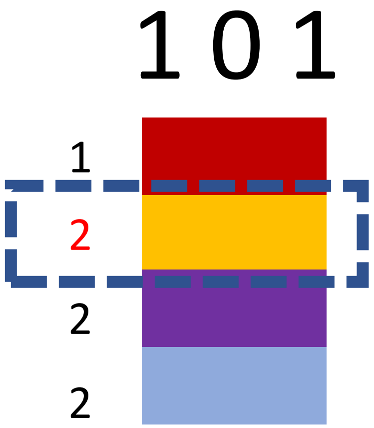

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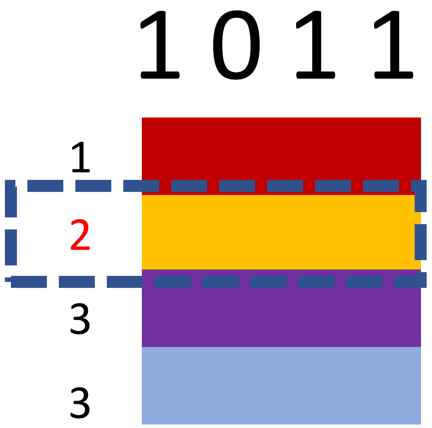

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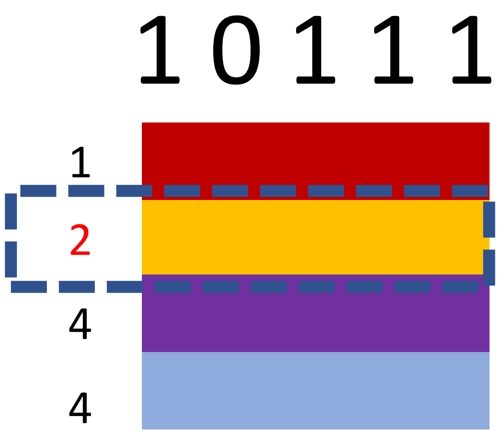

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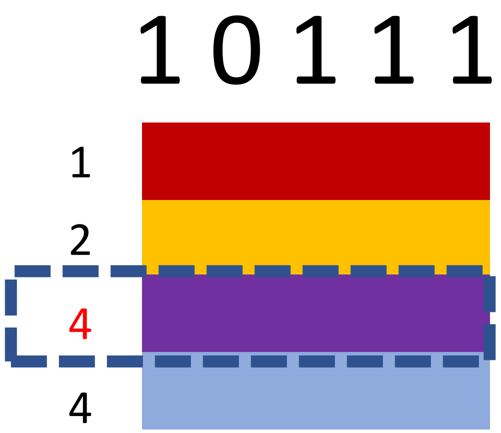

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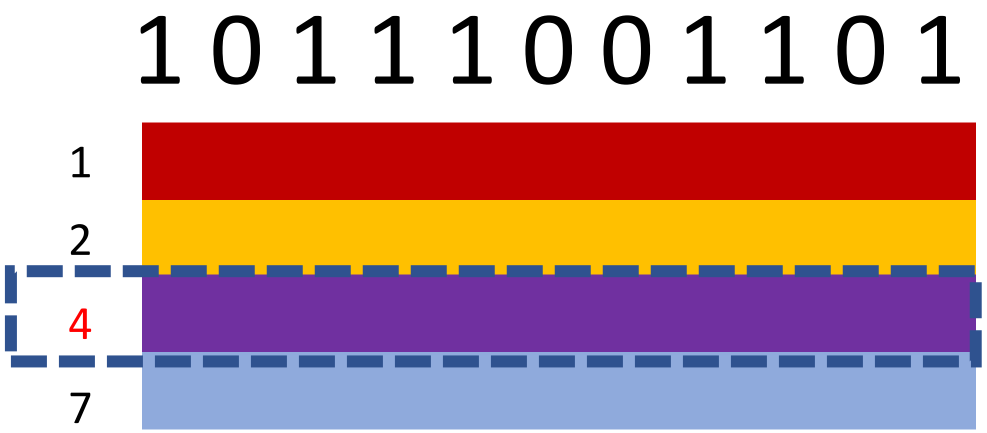


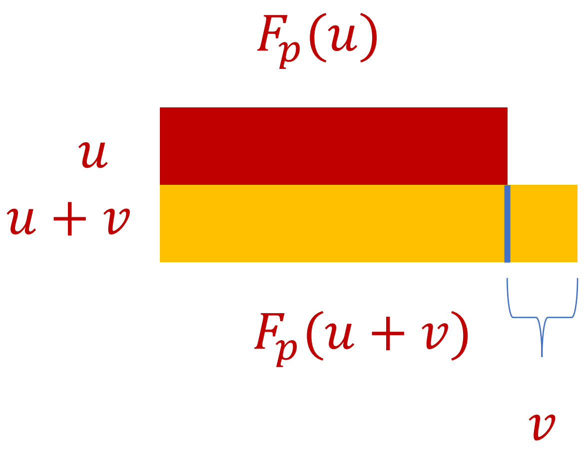
- Example: Number of ones in the stream (2-approximation)
- Number of ones stream is at least 4 and at most 8
- 4 is a good approximation

Summary

- Sketch switching for robust algorithms uses $\frac{1}{\epsilon^2}$ space each time F_p increases by $(1+\epsilon)$ and function increases $\frac{1}{\epsilon}$ times
- Smooth histogram for sliding window algorithms uses $\frac{1}{\epsilon^2}$ space each time F_p increases by $(1+\epsilon)$ and function increases $\frac{1}{\epsilon}$ times for $p \in (0,1)$
- Smooth histogram for sliding window algorithms uses $\frac{1}{\epsilon^2}$ space each time F_p increases by $(1+\epsilon^p)$ and function increases $\frac{1}{\epsilon^p}$ times for $p \in (1,2)$

Intuition

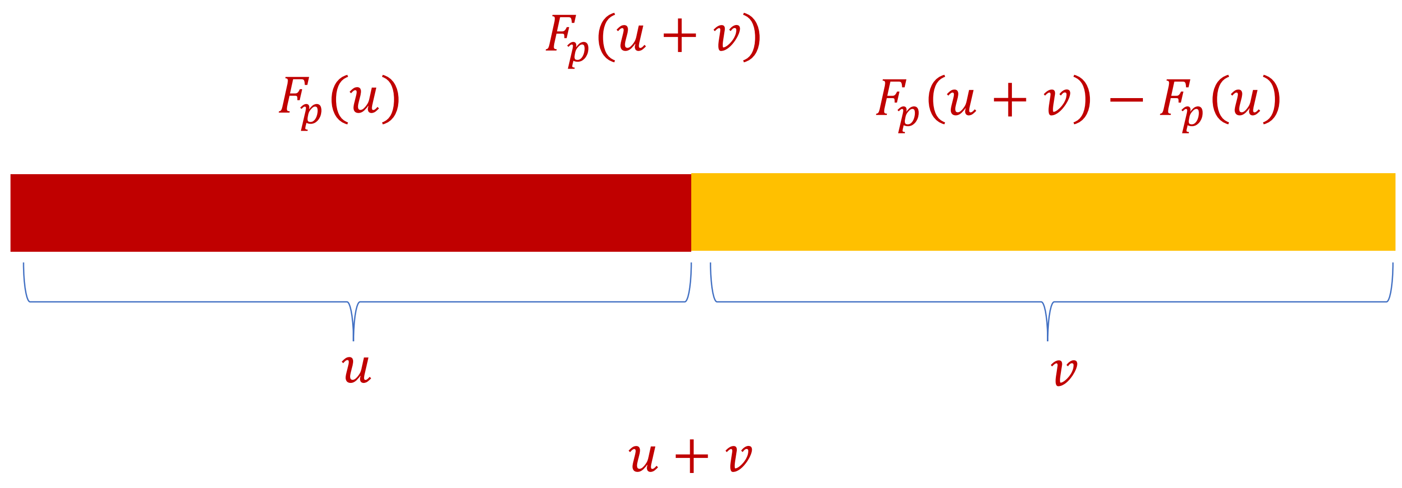
- Do we really need to pay $\frac{1}{\epsilon^2}$ space each time F_p increases by $(1+\epsilon)$?
- Only need constant factor approximation to $\epsilon F_p(u)$
- Only need constant factor approximation to $F_p(u+v) - F_p(u)$



$F_p(u+v) - F_p(u) = \epsilon F_p(u)$

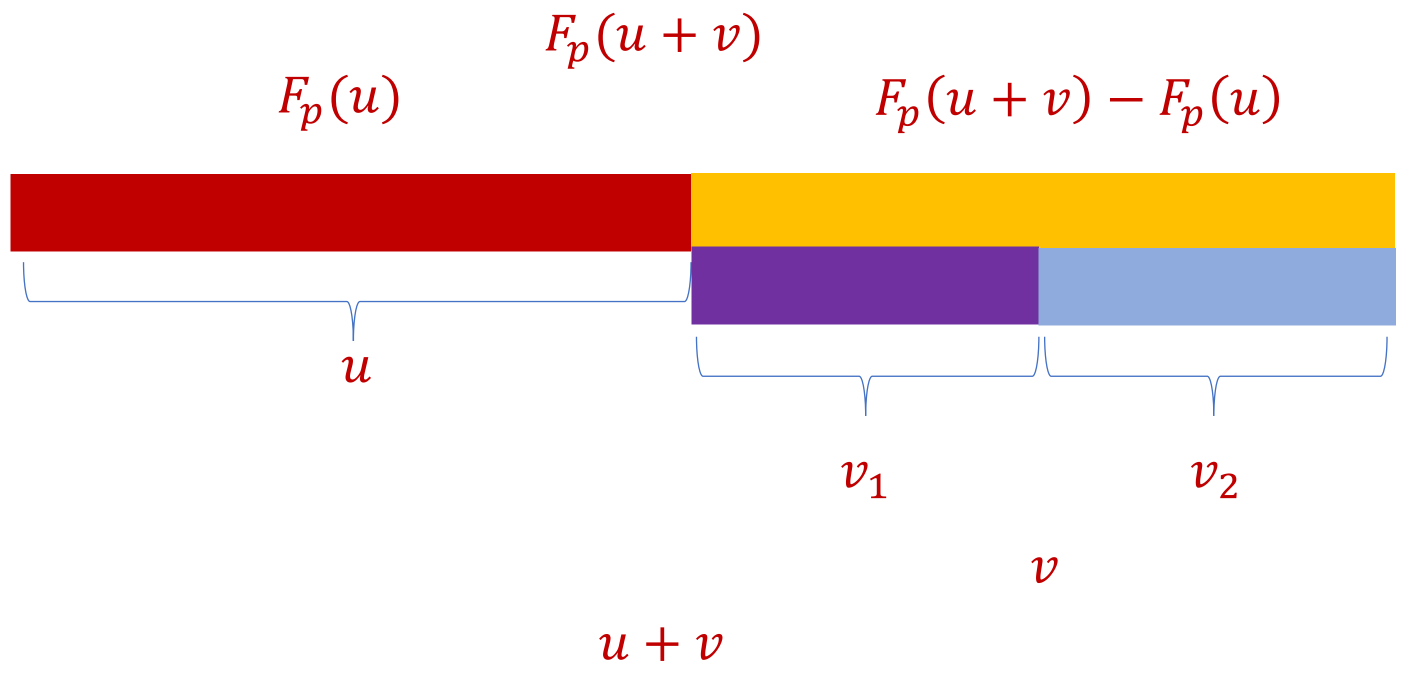
Sketch Stitching

- Suppose we want $F_p(u+v)$
- $F_p(u+v) = (F_p(u+v) - F_p(u)) + F_p(u)$



Sketch Stitching

- Suppose we want $F_p(u+v)$ and $v = v_1 + v_2 + \cdots + v_b$



Sketch Stitching

- Suppose we want $F_p(u+v)$ and $v = v_1 + v_2 + \cdots + v_b$
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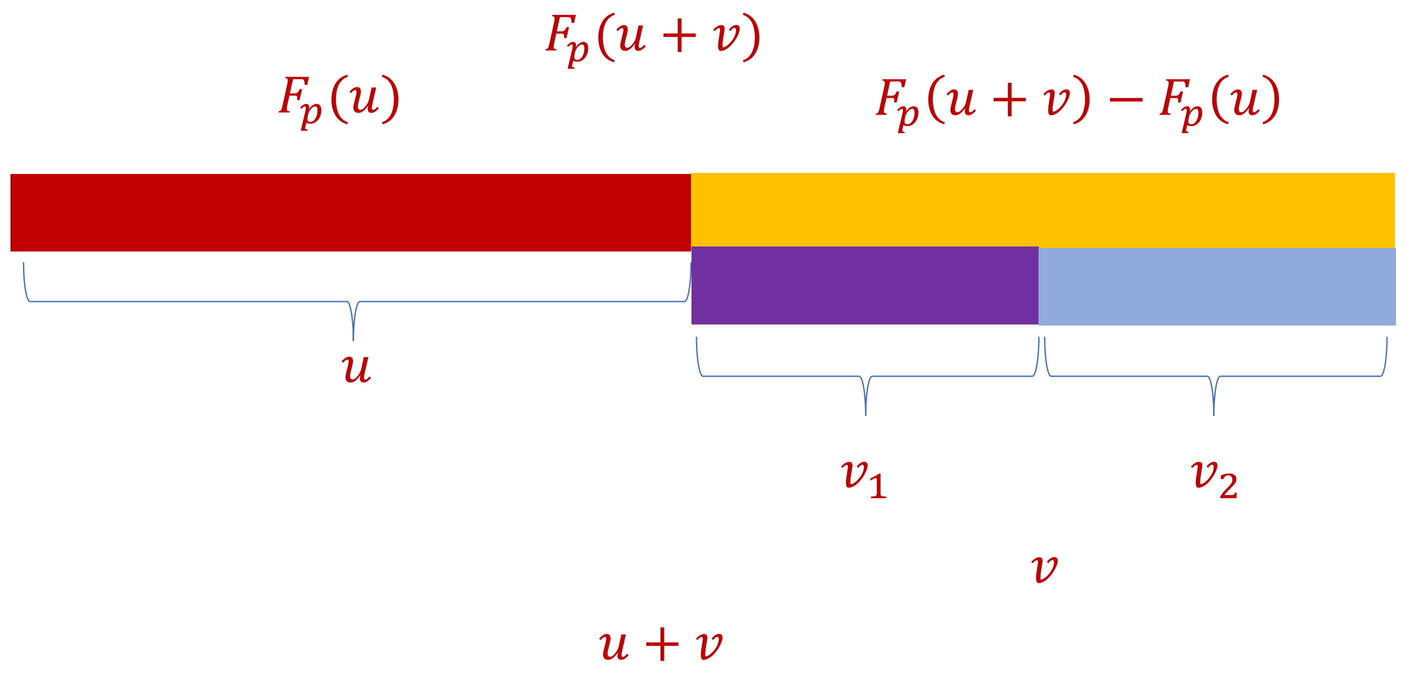


Sketch Stitching

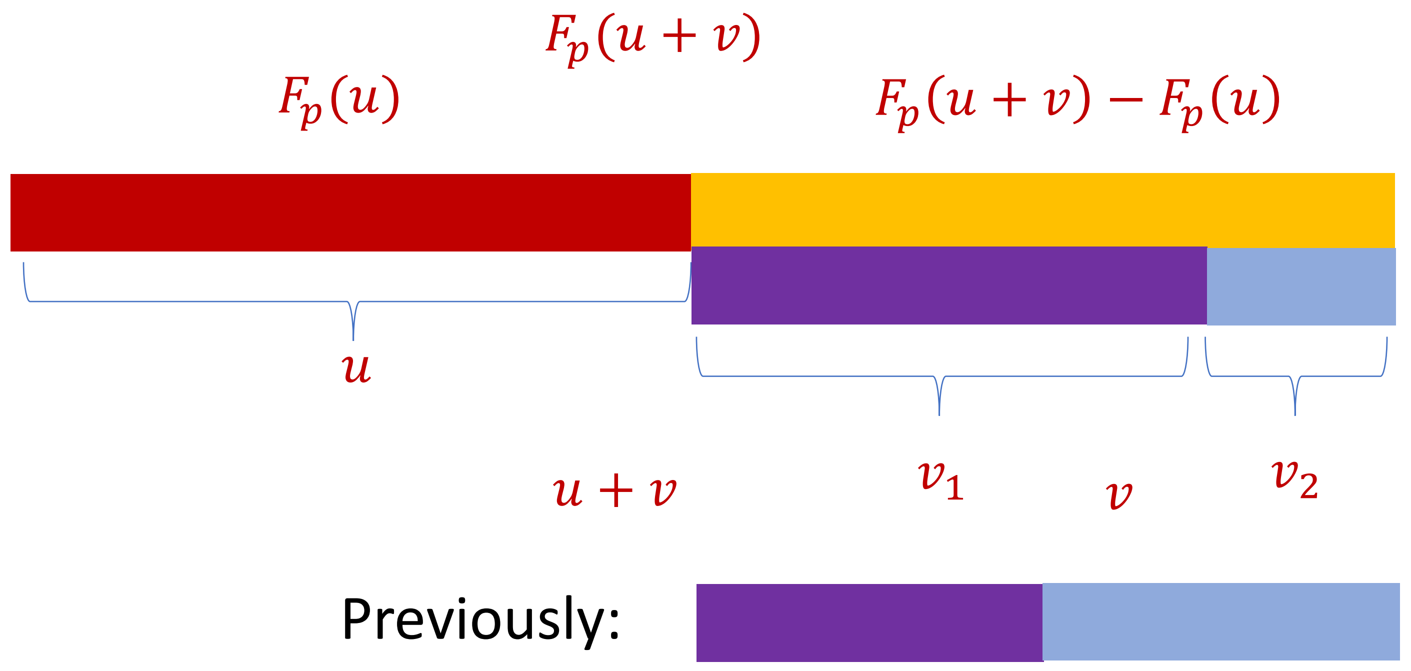
- Suppose we want $F_p(u+v)$ and $v = v_1 + v_2 + \cdots + v_b$
- $F_p(u+v) = (F_p(u+v) - F_p(u)) + F_p(u)$
- $F_p\,(u+v) = \left(F_p(u+v_1+\cdots+v_b) - F_p(u+v_1+\cdots+v_{b-1})\right) + \left(F_p(u+v_1+\cdots+v_{b-1}) - F_p(u+v_1+\cdots+v_{b-2})\right) + \cdots + \left(F_p(u+v_1) - F_p(u)\right) + F_p(u)$

Granularity Change

- Set each difference to be exponentially decreasing
- $F_p(u+v) = \Big(F_p(u+v_1+\cdots+v_b) - F_p(u+v_1+\cdots+v_{b-1})\Big) + \Big(F_p(u+v_1+\cdots+v_{b-1}) - F_p(u+v_1+\cdots+v_{b-2})\Big) + \cdots + \Big(F_p(u+v_1) - F_p(u)\Big) + F_p(u)$
- $F_p(u+v_1+\cdots+v_b) - F_p(u+v_1+\cdots+v_{b-1}) = \frac{1}{2^b} F_p(u)$

Granularity Change

- $F_p(u + v_1 + \cdots + v_b) - F_p(u + v_1 + \cdots + v_{b-1}) = \frac{1}{2^b} F_p(u)$

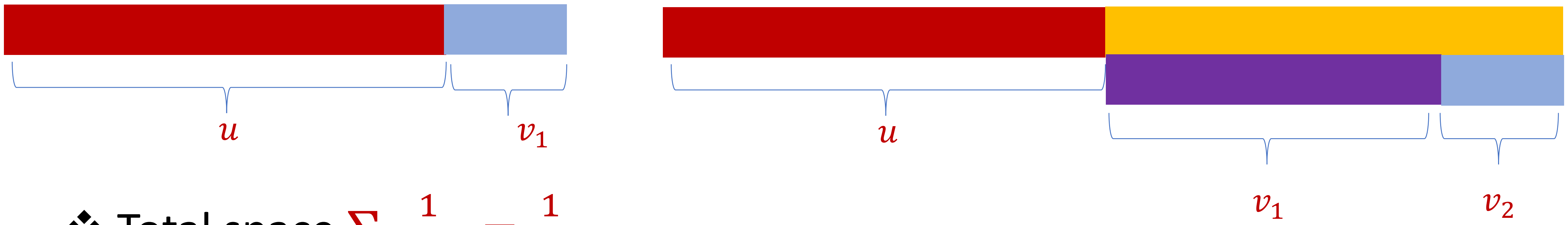


Granularity Change

- Set each difference to be exponentially decreasing
- $F_p\,(u+v) = \ \big(F_p(u+v_1+\cdots+v_b) - F_p(u+v_1+\cdots+v_{b-1})\big) + \big(F_p(u+v_1+\cdots+v_{b-1}) - F_p(u+v_1+\cdots+v_{b-2})\big) + \cdots + \big(F_p(u+v_1) - F_p(u)\big) + F_p(u)$
- $F_p(u+v_1+\cdots+v_b) - F_p(u+v_1+\cdots+v_{b-1}) = \frac{1}{2^b} F_p(u)$
- Just need $2^b \epsilon$-approximation to $F_p(u+v_1+\cdots+v_b) - F_p(u+v_1+\cdots+v_{b-1})$
- Hope is to use space $\frac{1}{2^{2b}\epsilon^2}$

Framework

- Algorithms simultaneously running for each granularity
- Want space $\frac{1}{2^{2b}\epsilon^2}$ for granularity $\frac{1}{2^b}F_p(u)$
- Need 2^b instances for granularity $\frac{1}{2^b}F_p(u)$



- Total space $\sum \frac{1}{2^b\epsilon^2} = \frac{1}{\epsilon^2}$

Format

- Part 1: Background
- Part 2: Framework
- Part 3: Difference Estimators

Questions?

Difference Estimator

- If $F_p(u+v) - F_p(u) = 2^b \epsilon F_p(u)$, does there exist algorithm that approximates the difference with space $\frac{1}{2^{2b}\epsilon^2}$?

- Definition: $F_p(u+v) - F_p(u) = \gamma F_p(u)$, output an estimate to the difference with additive approximation $\epsilon F_p(u)$

Difference Estimator

- Definition: $F_p(u+v) - F_p(u) = \gamma F_p(u)$, output an estimate to the difference with additive approximation $\epsilon F_p(u)$
- F is generally non-linear
- Ex: $F_p(u+v) = \frac{1}{\epsilon^4}$, $F_p(u+v) - F_p(u) = 1$
- $(1+\epsilon)$ approximations to $F_p(u+v)$ and $F_p(u)$ give multiplicative approximation to the difference but use space $\frac{1}{\epsilon^2}$
- Constant factor approximations to $F_p(u+v)$ and $F_p(u)$ do not give additive approximation $\epsilon F_p(u)$ to the difference

Our Results: Difference Estimators

- Space $\tilde{O}\left(\frac{\gamma}{\epsilon^2}\log n\right)$ algorithm for F_0
- Space $\tilde{O}\left(\frac{\gamma^{2/p}}{\epsilon^2}\log n\right)$ algorithm for F_p with $p \in (0,2]$
- Space $\tilde{O}\left(\frac{\gamma}{\epsilon^2}n^{1-2/p}\right)$ algorithm for F_p with integer $p > 2$

F_2 Difference Estimator

- Definition: $F_2(u+v) - F_2(u) = \gamma F_2(u)$, output an estimate to the difference with additive approximation $\epsilon F_2(u)$
- $F_2(u+v) - F_2(u) = \langle u+v, u+v \rangle - \langle u, u \rangle = 2\langle u, v \rangle + \langle v, v \rangle^2$
- Inner product property: $(1+\epsilon)$ -approximations to $\|u\|_2$ and $\|v\|_2$ gives an $\epsilon \|u\|_2 \|v\|_2$ additive approximation to $\langle u, v \rangle$
- $\gamma F_2(u) \geq \langle v, v \rangle^2$ implies $2\langle u, v \rangle \leq 2\|u\|_2 \|v\|_2 \leq 2\sqrt{\gamma}\, F_2(u)$
- Just need $\frac{\epsilon}{\sqrt{\gamma}}$ multiplicative approximation: $\tilde{O}\left(\frac{\gamma}{\epsilon^2} \log n\right)$ space!

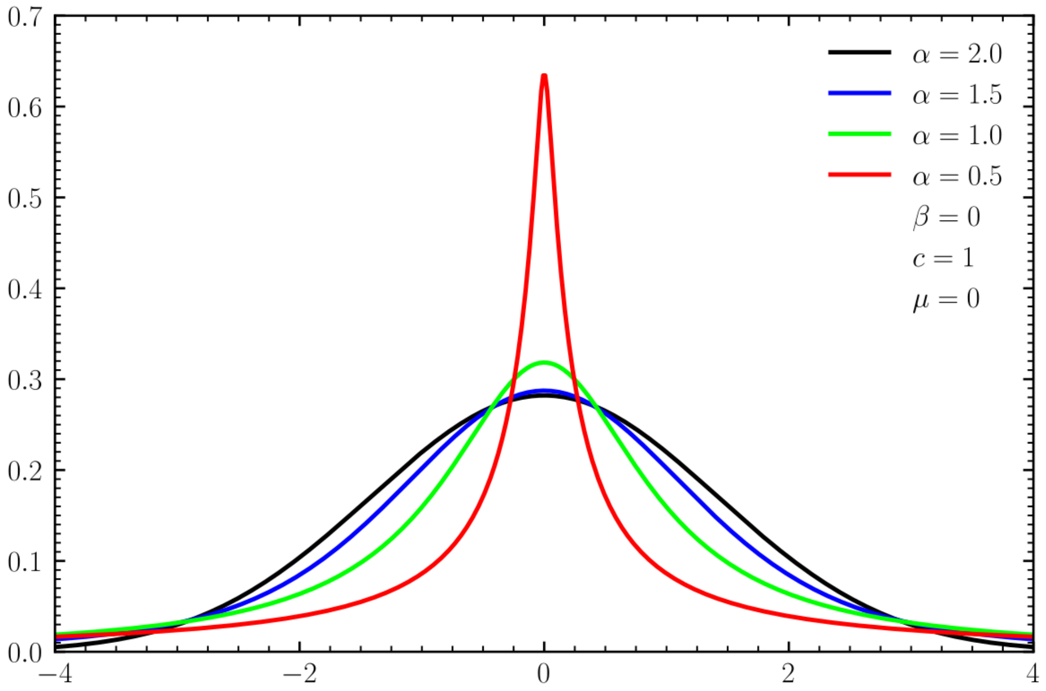
F_2 Difference Estimator

- Difference estimator: Maintain $\left(1+\frac{\epsilon}{\sqrt{\gamma}}\right)$-approximations to $F_2(u+v)$ and $F_2(u)$ using AMS sketch
- For $F_2(u+v)-F_2(u)=\gamma F_2(u)$, difference of the outputs is an additive approximation $\epsilon F_2(u)$ to $F_2(u+v)-F_2(u)$

Challenges for Difference Estimators

- F_p difference estimator: Use p-stable random variables for $p \leq 2$?
- How to use approach of [BlasiokDingNelson17]?
- $\langle Z_p, f \rangle$ where Z_p has entries drawn from p-stable distribution

$$p(x) \sim \frac{1}{1 + |x|^{p+1}}$$



Challenges for Difference Estimators

- $Z = median\langle Z_p, f\rangle$
- How to analyze median of each estimate of $F_p(u+v) - F_p(u)$?

- Use Li's geometric mean algorithm [Li08]
- Take the geometric mean of 3 inner products $\langle Z_p, f\rangle$
- Take the average of $O\left(\frac{1}{\epsilon^2}\right)$ geometric means

Challenges for Difference Estimators

- Difference estimator: Maintain $\left(1+\frac{\epsilon}{\gamma^{1/p}}\right)$-approximations to $F_p(u+v)$ and $F_p(u)$ using Li's geometric mean estimator
- $A(u+v)-A(u) \sim \langle p_1, v\rangle^{p/3}\langle p_2, v\rangle^{p/3}\langle p_3, v\rangle^{p/3} +$
  $\langle p_1, u\rangle^{p/3}\langle p_2, v\rangle^{p/3}\langle p_1, v\rangle^{p/3} +$
  $\langle p_1, u\rangle^{p/3}\langle p_2, u\rangle^{p/3}\langle p_1, v\rangle^{p/3} + \cdots$

- Each summand has $\langle p_1, v\rangle^{p/3}$ term, which has much smaller variance

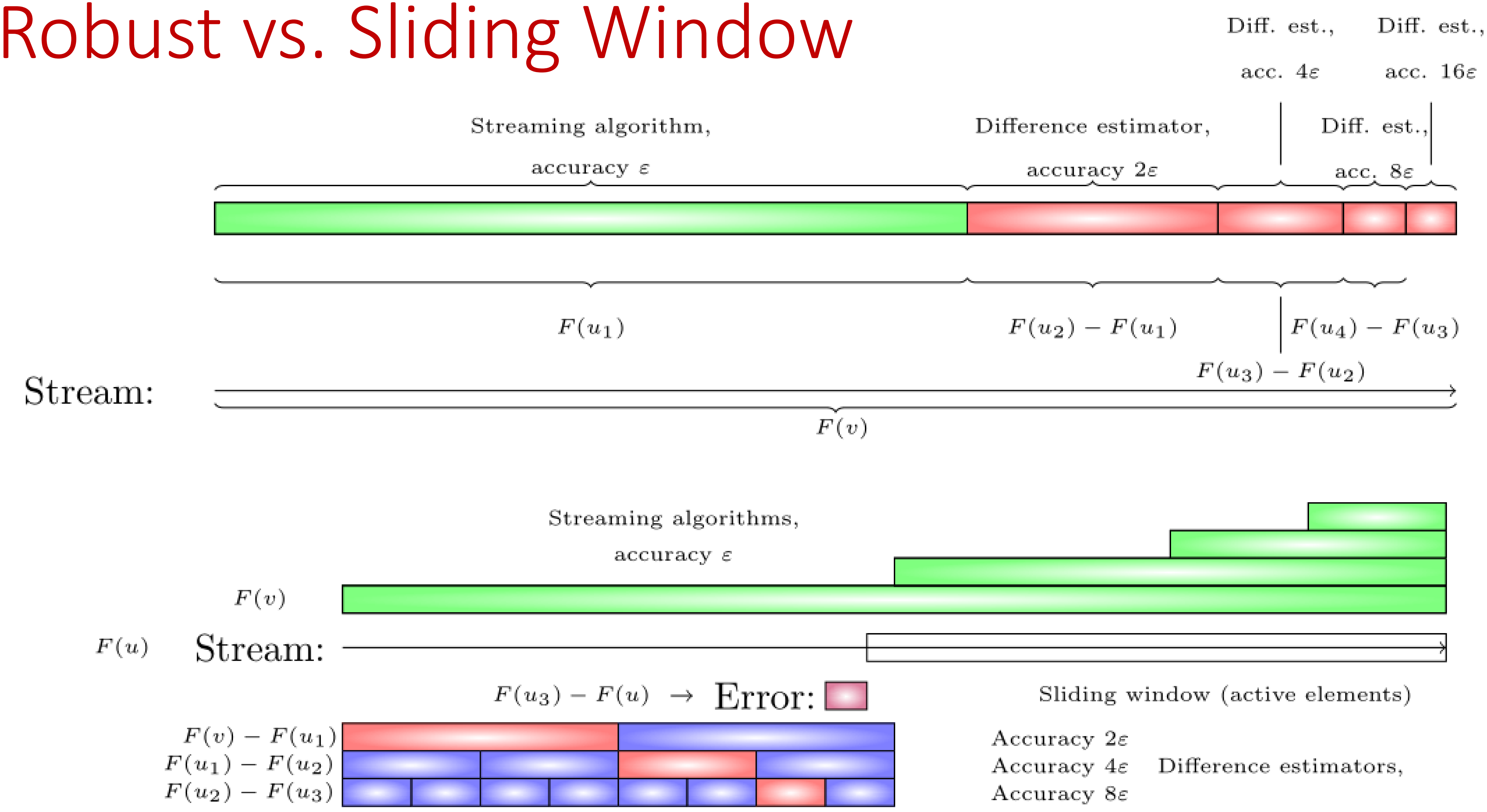
Challenges for Difference Estimators

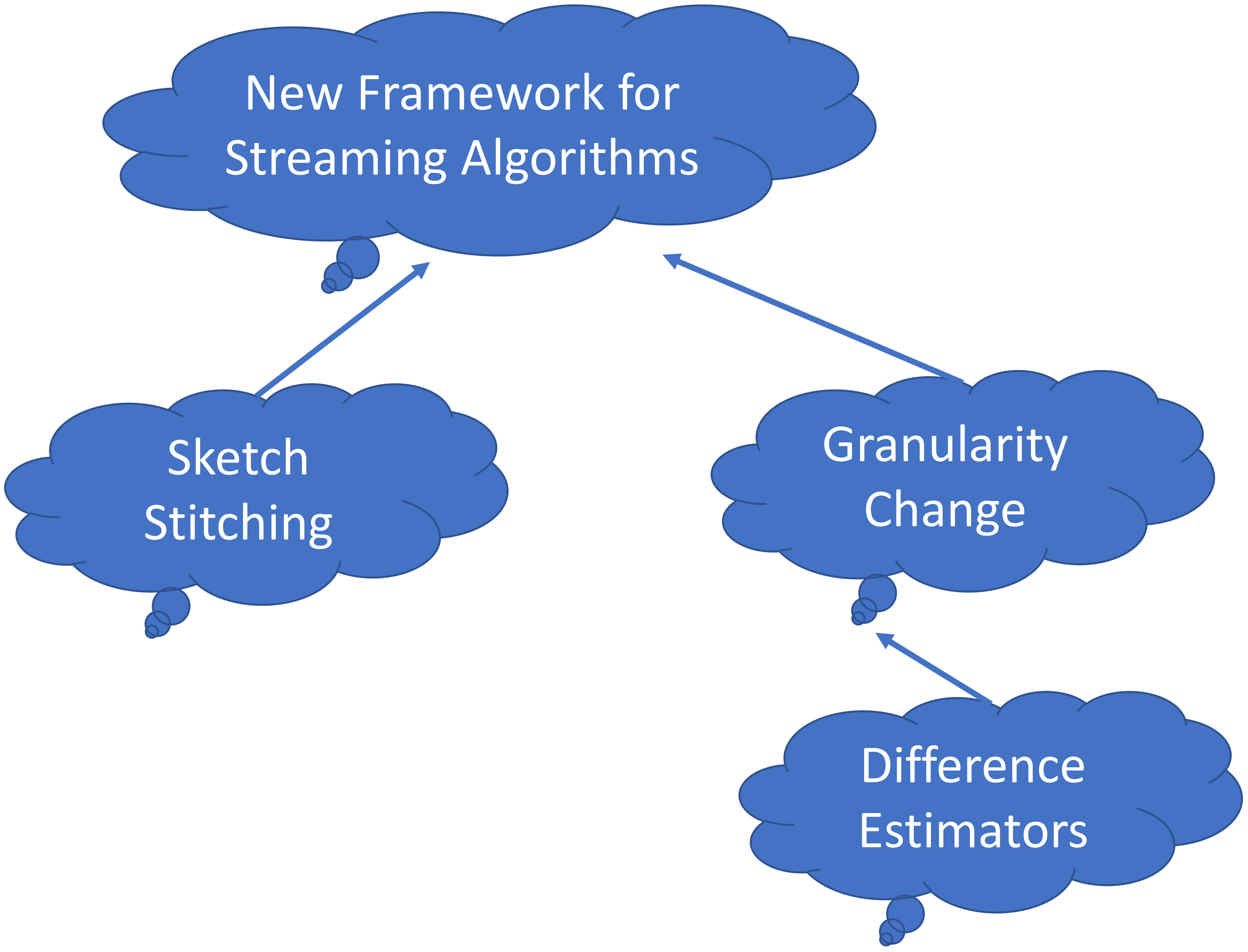
- F_p difference estimator: Generalization of inner products for $p > 2$?
- Variance can be much larger!
- Use heavy-hitter algorithm to explicitly track “heavy” elements
- Use L_2 sampling algorithm with $n^{1-2/p}$ buckets to sample “light” elements

Challenges for Difference Estimators

- Use known structural results from chaining to remove $\log n$ factor in difference estimator
- Avoids typical Chernoff + union bound argument by considering the expected supremum of a process, "strong tracking"
- Use suffix argument to remove $\log n$ factor in framework
- Adaptation to sliding window model

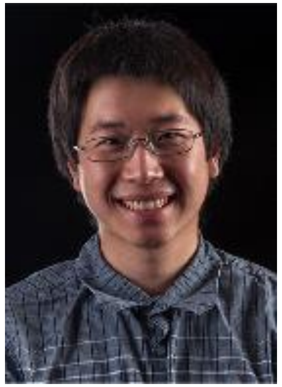
Robust vs. Sliding Window




New Framework for Streaming Algorithms
Sketch Stitching
Granularity Change
Difference Estimators

WALDO 2021: Save the Date!

- Workshop on Algorithms for Large Data Online
- Streaming, sketching, linear algebra, property testing, learning
- Virtual format, August 23-25



Future Directions

- Other applications of difference estimators?
- Tighter bounds for difference estimators
- Difference estimators for general $p > 2$?
- Other adversarially robust algorithms?