Fast Fourier Sparsity Testing

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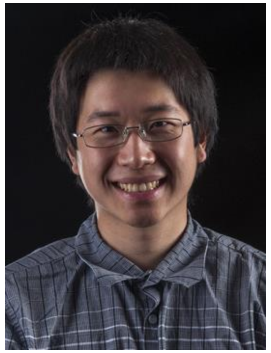


The Alan Turing Institute

Carnegie Mellon University

Fourier Expansion

For $S \subseteq [n]$, the characteristic function $\chi_S(x): \{-1,+1\}^n \rightarrow \{-1,+1\}$ is defined as $\chi_S(x) = \prod_{i \in S} x_i$

The Fourier expansion of a function $f: \{-1,+1\}^n \rightarrow \mathbb{R}$ is the unique linear combination of multilinear polynomials:

$$f(x_1, \ldots, x_n) = \sum_{S \subseteq [n]} \hat{f}(S) \chi_S(x)$$

Then we have the Fourier coefficient $\hat{f}(S) = \langle f, \chi_S \rangle = E[f(x)\chi_S(x)]$

Fourier Sparsity

$$f(x_1, x_2, x_3) = Maj_3(x_1, x_2, x_3) = \frac{1}{2}x_1 + \frac{1}{2}x_2 + \frac{1}{2}x_3 - \frac{1}{2}x_1 x_2 x_3$$

$$f(x_1, x_2, x_3, x_4, x_5, x_6) = 6 + 6x_1 x_2$$
$$= 3(x_1 + x_2)^2 - x_3^3 x_4 + 2x_3 x_4 x_5^2 - x_3 x_4^3 x_6^4$$

A function is s-sparse if it has at most s nonzero Fourier coefficients

Why Fourier Sparsity?

Fourier sparsity has applications in coding theory [GoldreichLevin89, AkaviaGoldwasserShafra03], learning theory [KushilevitzMansour93, LinialMansourNisan93], communication complexity [ShiZhang09]

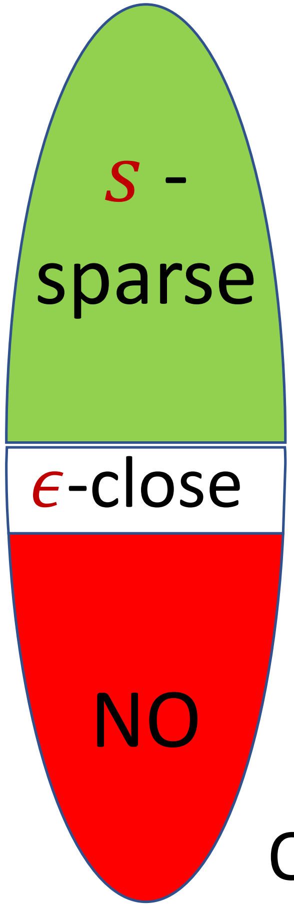
If a function is known to be s-sparse, more efficient algorithms can often be run, e.g. sparse Fourier transform [HIKP12]

Property Tester

Testing sparsity of Boolean functions under Hamming distance

- [GOSSW11]
  - Non-tolerant test
  - Complexity $O\left(s^{14}\log s+\frac{s^6}{\epsilon^2\log s}\right)$
  - Reduction to testing under ℓ_2
  - Lower bound $\Omega(\sqrt{s})$
- [WimmerYoshida13]
  - Tolerant test
  - Complexity $\mathrm{poly}\left(s,\frac{1}{\epsilon}\right)$

Our results give a tolerant test with almost quadratic improvement on [GOSSW11]



$\Rightarrow$ Accept with probability $\geq \frac{2}{3}$

$\Rightarrow$ Don't care

$\Rightarrow$ Reject with probability $\geq \frac{2}{3}$

Query model: $(x, f(x))$, for any $x \in \{-1,+1\}^n$

Our Contributions

Upper bound: Algorithm that makes $O\left(\frac{s}{\epsilon^4}\log\frac{1}{\epsilon}\right)$ non-adaptive queries to a (normalized) function f and approximates the ℓ_2^2 distance from f to the set of s-sparse functions. Translates to a tester.

Lower bound: Any algorithm that distinguishes whether f is s-sparse or at least $\frac{1}{3}$ - far from s-sparse in ℓ_2^2 distance requires $\Omega(\sqrt{s})$ queries

Random Hashing [FGKP09]

If a subspace H of $\{-1,+1\}^n$ is drawn randomly from subspaces of codimension d, then $\Pr[b \in a + H] = \frac{1}{2^d}$ for distinct $a, b \neq 0$

For an element $a \in H^\perp$, the projected function $f|_{a+H}(z) = E_{x \in H^\perp}[f(x+z)\chi_a(x)]$ for each $z \in \{-1,+1\}^n$

Poisson Summation Formula: $\hat{f}|_{a+H}(\alpha) = \hat{f}(\alpha)$ if $\alpha \in a + H$ and 0 otherwise

Define the total energy of $f|_{a+H}$ as $\sum_{\alpha \in a+H} \hat{f}(\alpha)^2 = \left\|\hat{f}|_{a+H}\right\|_2^2$

Testing s-Sparsity

 $= O(s^2) \Rightarrow$ 

$\sum_{\alpha \in a+H} \hat{f}(\alpha)^2 = E[\chi_a(z) f(x) f(x+z)]$, where $x \in \{-1,+1\}^n, z \in H^{\perp}$ [GOSSW11]

The set of queries $\{f(x+z)\}_{x \in H^{\perp}}$ can be used to compute $f|_{a+H}(z)$ for each of the cosets $a+H$ simultaneously

Algorithm

Draw subspace H of codimension $d = \log \frac{2s}{\epsilon^4}$ at random

Draw $O\left(\frac{s}{\epsilon^4}\right)$ pairs $(x, x + z)$, where $x \in \{-1, +1\}^n, z \in H^\perp$

Estimate the energy y_{a+H} for each $a \in H^\perp$

$$y_{a+H} \mathrel{+}= O\left(\frac{\epsilon^4}{s}\right) \chi_a(z) f(x) f(x+z)$$

Repeat $\ell = O\left(\log \frac{1}{\epsilon}\right)$ times, take the largest sum of the energies in s buckets

$$\xi = \max_{S \in H^\perp, |S| = s} \sum_{a \in S} \operatorname{median}\left(y_{a+H}^{(1)}, \dots, y_{a+H}^{(\ell)}\right)$$

Analysis

Top s coefficients may collide in a bucket

Noise from non top s coefficients

Hashing error: Let $y_1 \geq \cdots \geq y_s$ be the energies of the top s buckets and $f_1 \geq \cdots \geq f_s$ be the energies of the top s Fourier coefficients. Then with probability at least 15/16, the hashing error $\sum y_i - f_i \leq 5\epsilon^2$

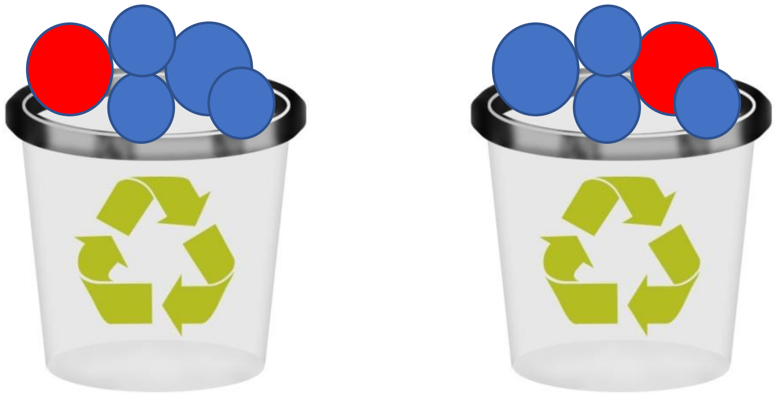
Estimation error: Let $y_1 \geq \cdots \geq y_s$ be the energies of the top s buckets and $\widehat{y}_i$ be the estimate of y_i. Then $E_H[\sum_{i=1}^{s} |\widehat{y}_i - y_i|^2] \leq \epsilon^2$

Putting it all together

Let S^* be the set of s buckets that maximize the estimated energies

Define h to be f with only the s Fourier coefficients that are the largest Fourier coefficient in each bucket, but their energies is the energy of the entire bucket

$\xi = \|h\|_2^2$



Putting it all together

Define f^* to be f with only the largest s Fourier coefficients

$$\left|\xi - \|f^*\|_2^2\right| = \left|\|h\|_2^2 - \|f^*\|_2^2\right| \leq 2\|f^* - h\|_2$$

Define g to be f with only the s Fourier coefficients that are the largest Fourier coefficient in each bucket

Then $\|f^* - h\|_2 \leq \underbrace{\|f^* - g\|_2}_{\text{Hashing error}} + \underbrace{\|g - h\|_2}_{\text{Estimation error}} = O(\epsilon)$

Future Work?

Improve upper or lower bounds

Extensions to other domains (line, hypergrid)

Other properties that can be tested in ℓ_2?

NSF


thank you
danke
謝謝
ngiyabonga
teşekkür ederim
gracias
merci
obrigado
dziękuję
bedankt
спасибо
go raibh maith agat
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terima kasih
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bayarlalaa
gracie
köszönöm
nanni
nandri
didi madloba
rahmat
tau

Hashing Error

Hashing error: Let $y_1 \geq \cdots \geq y_s$ be the energies of the top s buckets and $f_1 \geq \cdots \geq f_s$ be the energies of the top s Fourier coefficients. Then with probability at least 15/16, the hashing error is at most $5\epsilon^2$ (Essentially just avoid collisions)

Define y_i^* as the largest Fourier coefficient hashing into bucket i

$$\sum y_i - f_i = \sum y_i - y_i^* + \sum y_i^* - f_i \leq \sum y_i - y_i^*$$

$E[\sum y_i - y_i^*] \leq \sqrt{\frac{2s}{2^d}}$, $\mathrm{Var}(\sum y_i - y_i^*) \leq \frac{2}{2^d}$ by Cauchy-Schwartz and Jensen, result holds by Chebyshev

Estimation Error

Estimation error: Let $y_1 \geq \cdots \geq y_s$ be the energies of the top s buckets and $\widehat{y_i}$ be the estimate of y_i. Then $E_H[\sum_{i=1}^{s} |\widehat{y_i} - y_i|^2] \leq \epsilon^2$

For a fixed $j \in [\ell]$, $E_H\left[\sum_{i=1}^{s} \left|y_{i,j} - y_i\right|^2\right] \leq \frac{\epsilon^4}{s}$ (number of samples)

By Markov and counting, $\Pr[|\widehat{y_i} - y_i|^2 \geq \eta] \leq \left(\frac{2e\epsilon^4}{s\eta}\right)^{\ell/2}$

Result follows from integrating probability density function and Jensen