Sensitivity Analysis of the Maximum Matching Problem

Yuichi Yoshida

Samson Zhou

Average Sensitivity for Graph Algorithms

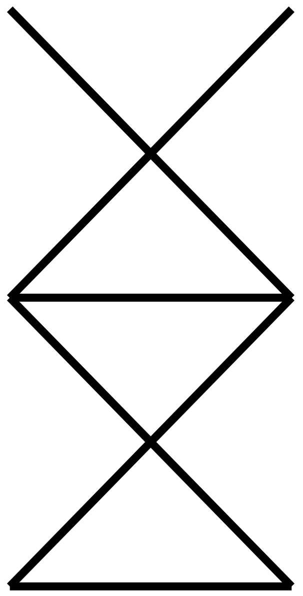
- Measure of how much a discrete algorithm changes when the input changes
- For a deterministic graph algorithm A on a graph G with n vertices, the average sensitivity is defined as

$$\mathop{\mathrm{E}}_{e \sim E(G)} [|A(G) \triangle A(G - e)|]$$

where $\triangle$ is the symmetric difference

Average Sensitivity for Graph Algorithms

- Example: Greedy algorithm for maximal matching on graph with 6 nodes



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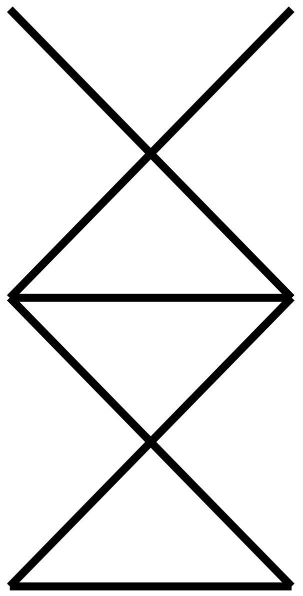

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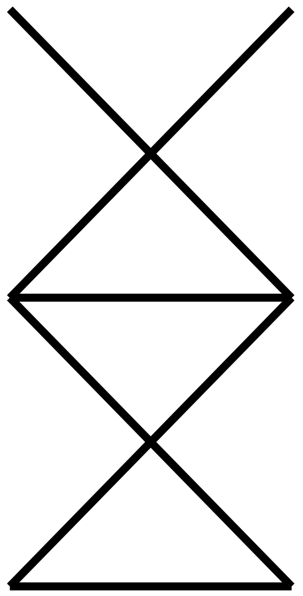

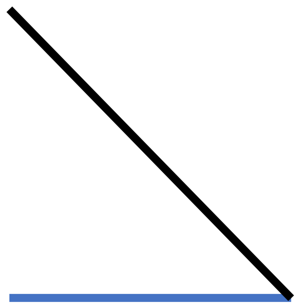

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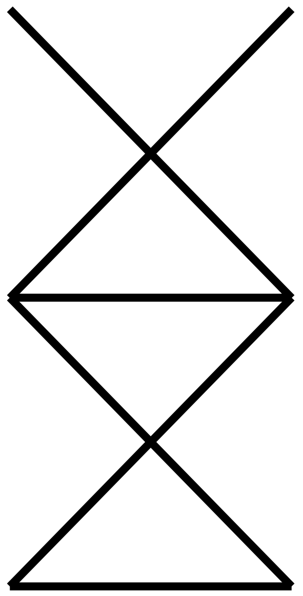

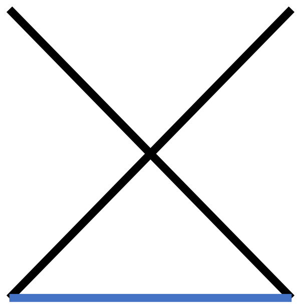

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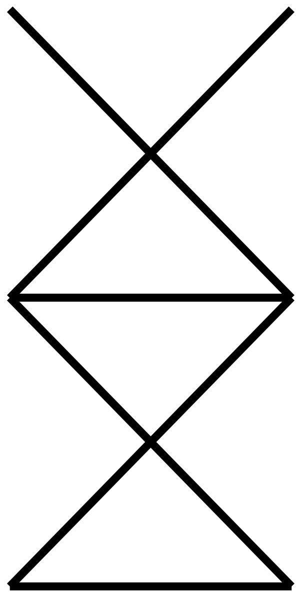

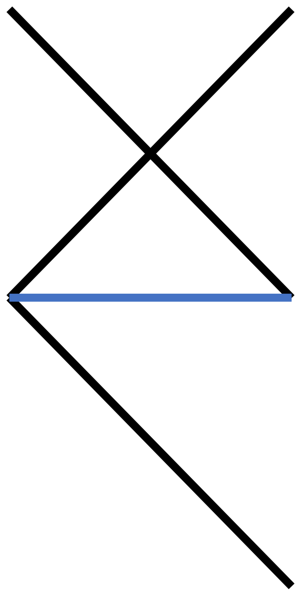

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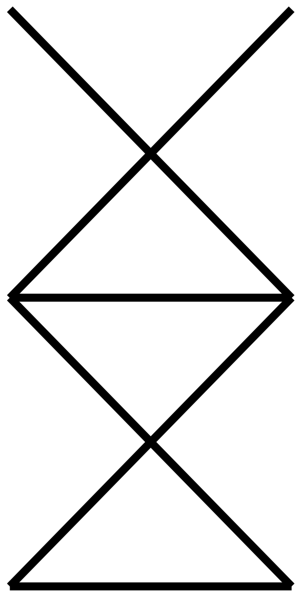

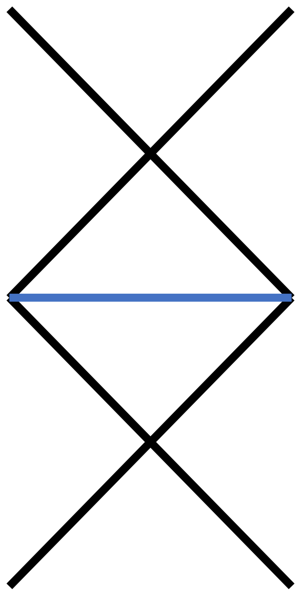

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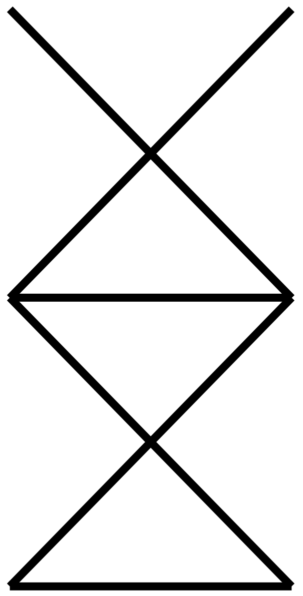

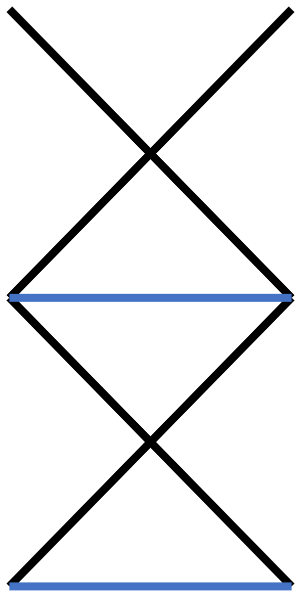

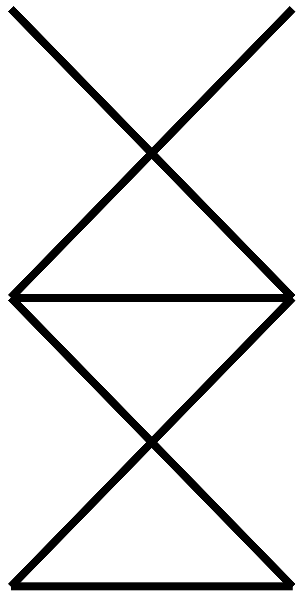

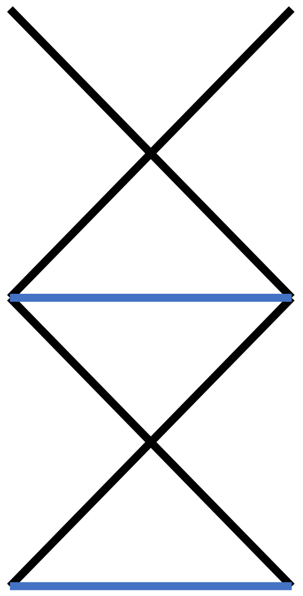

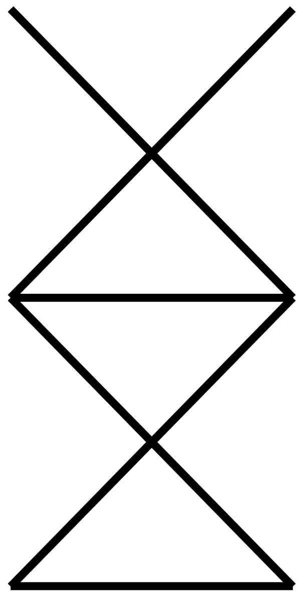

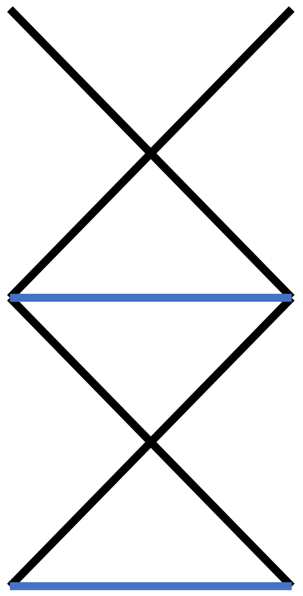

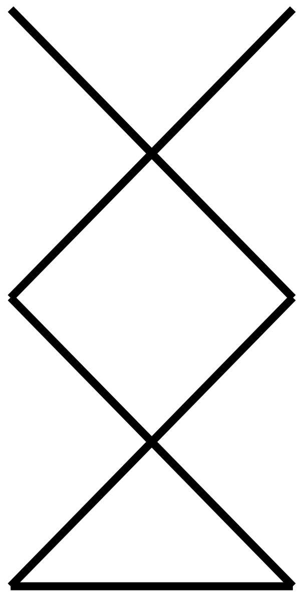

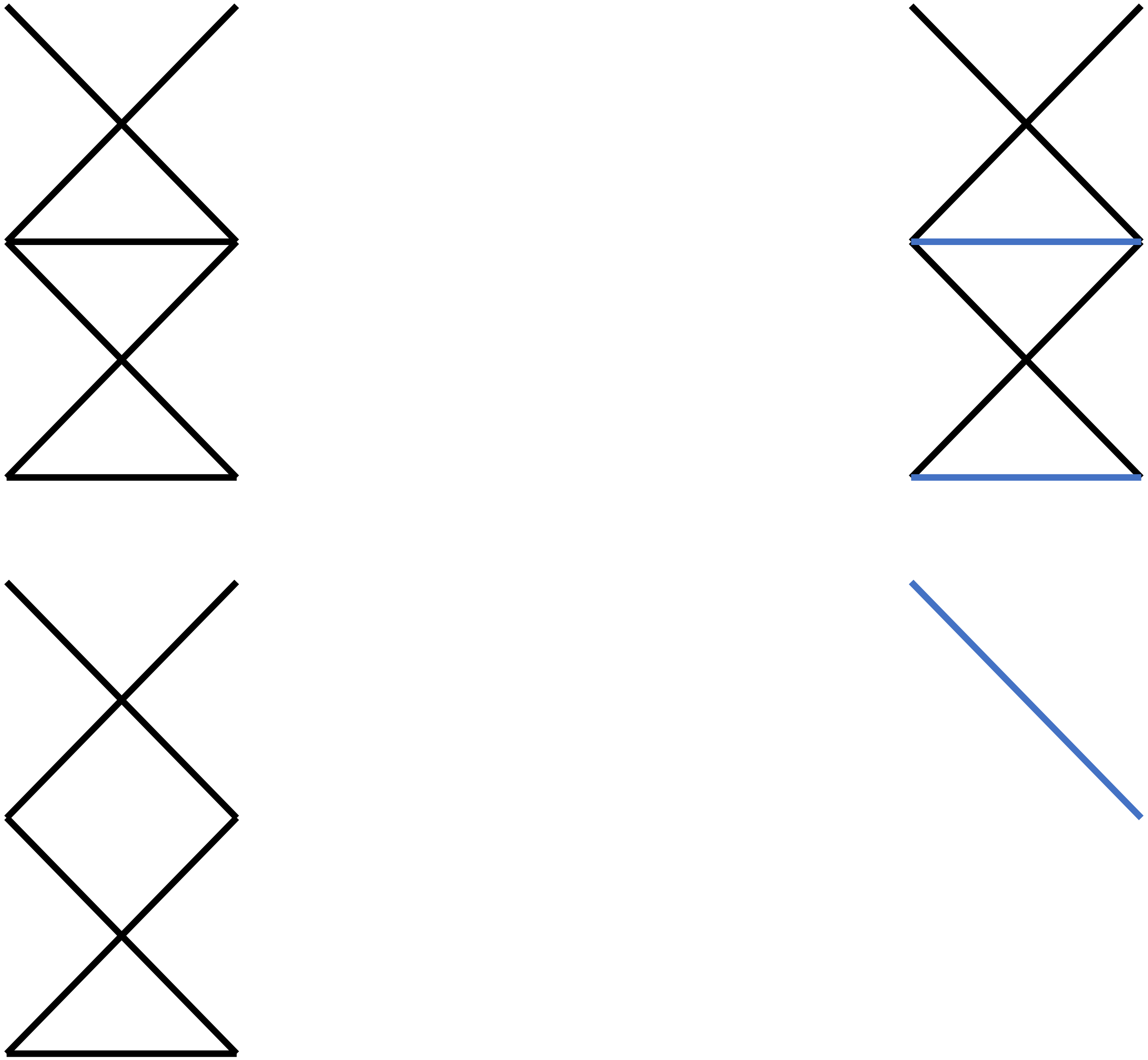

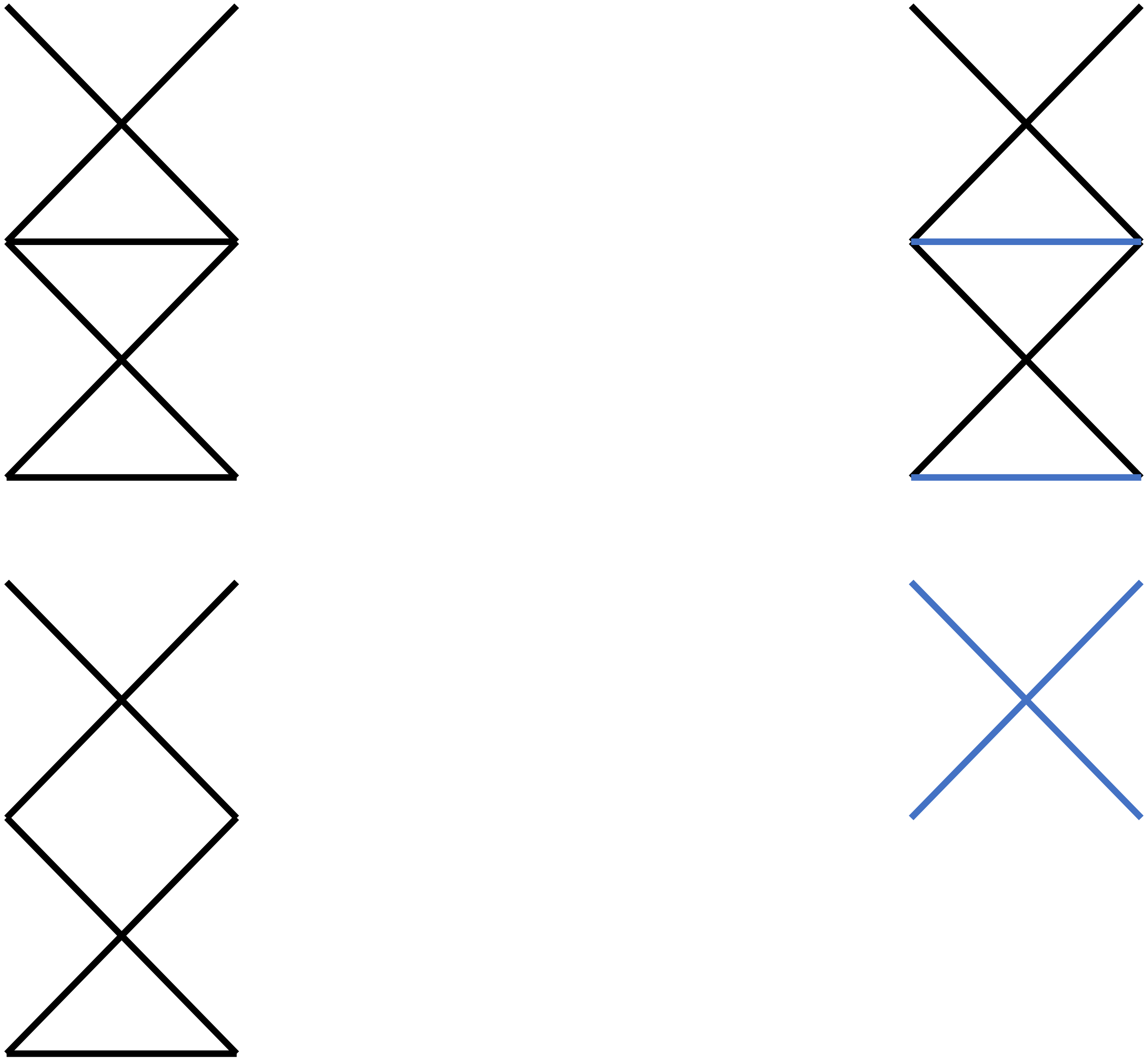

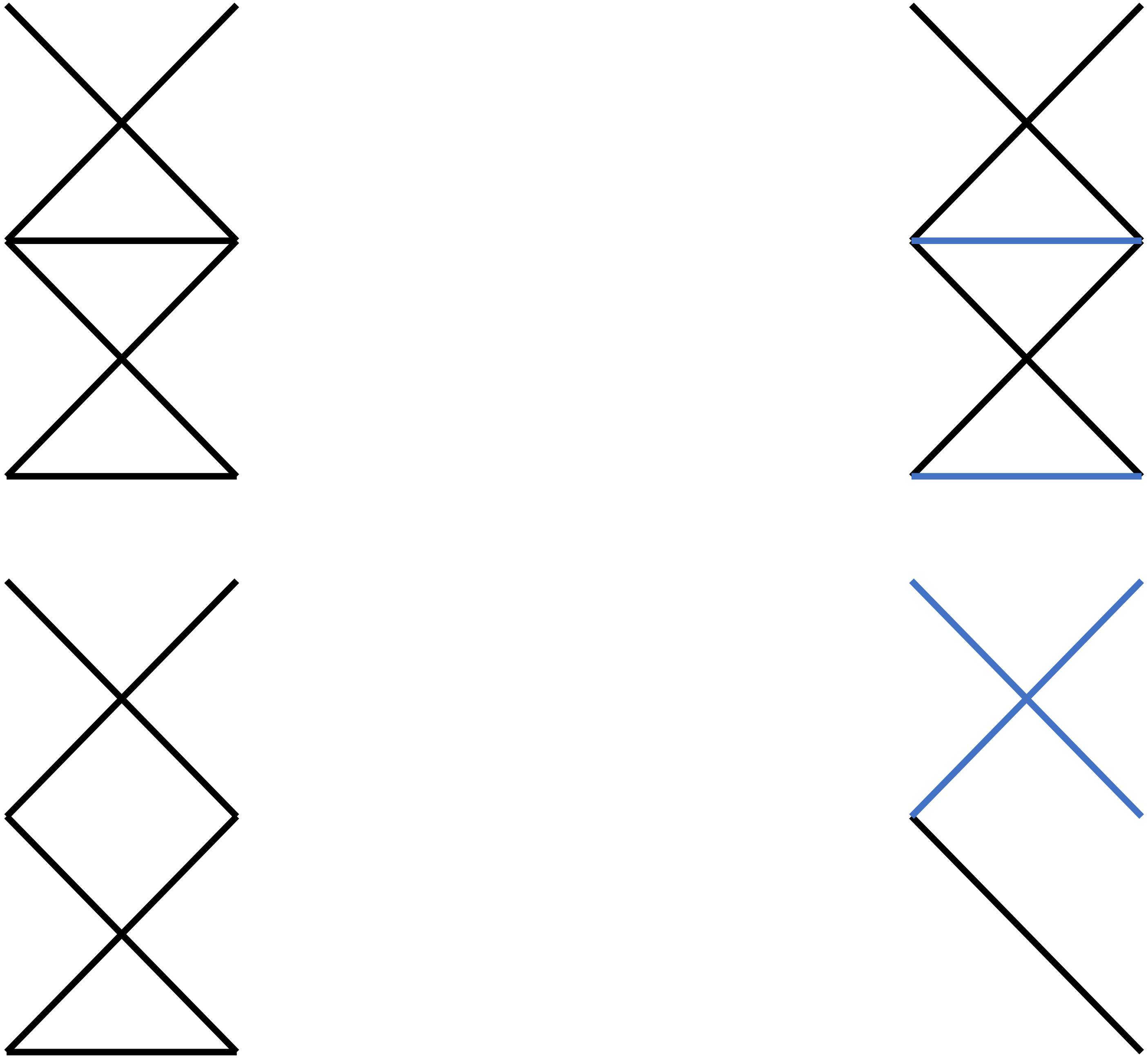

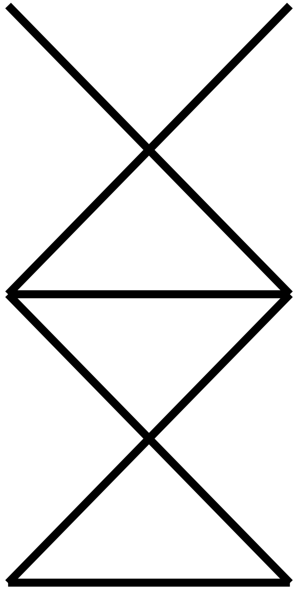

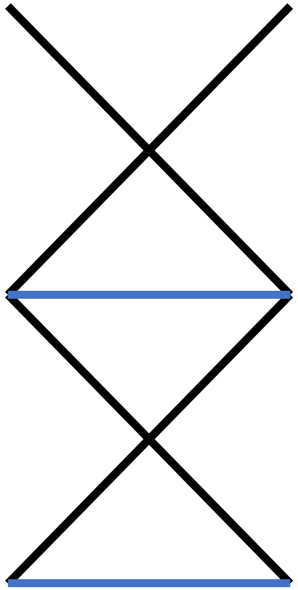

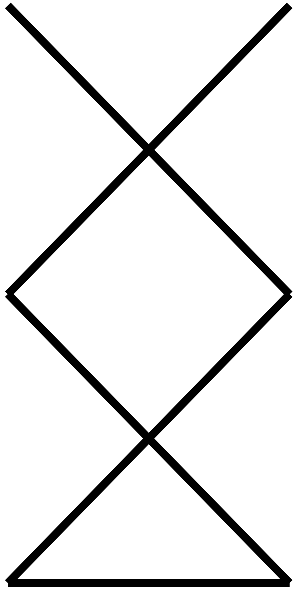

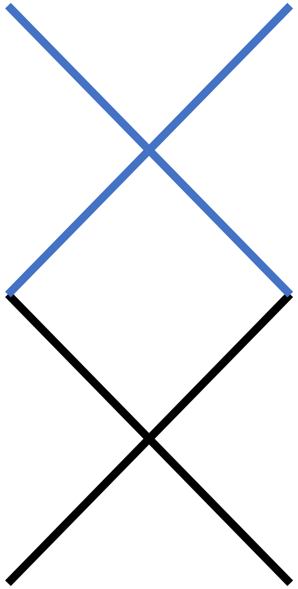

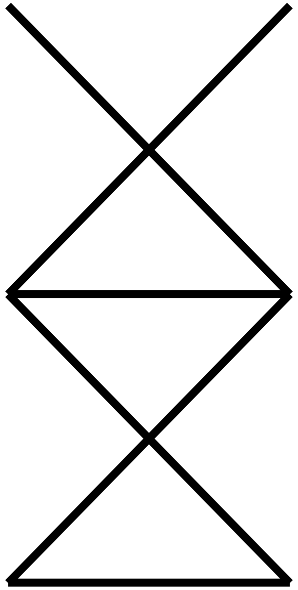
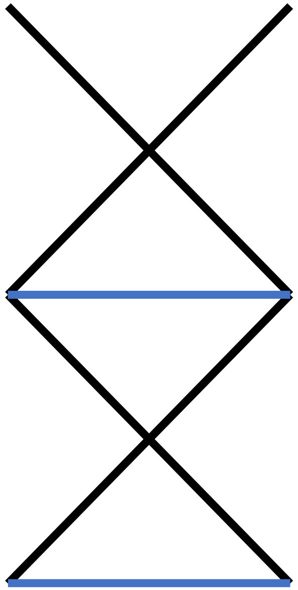
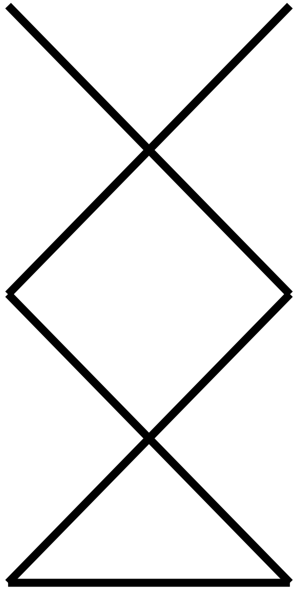
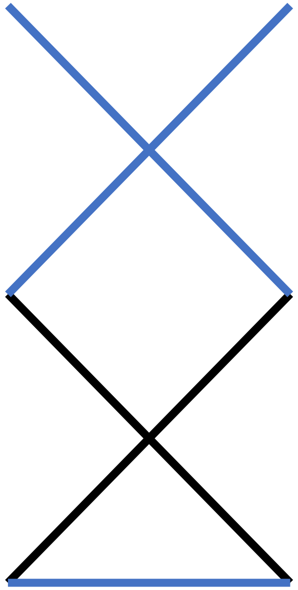

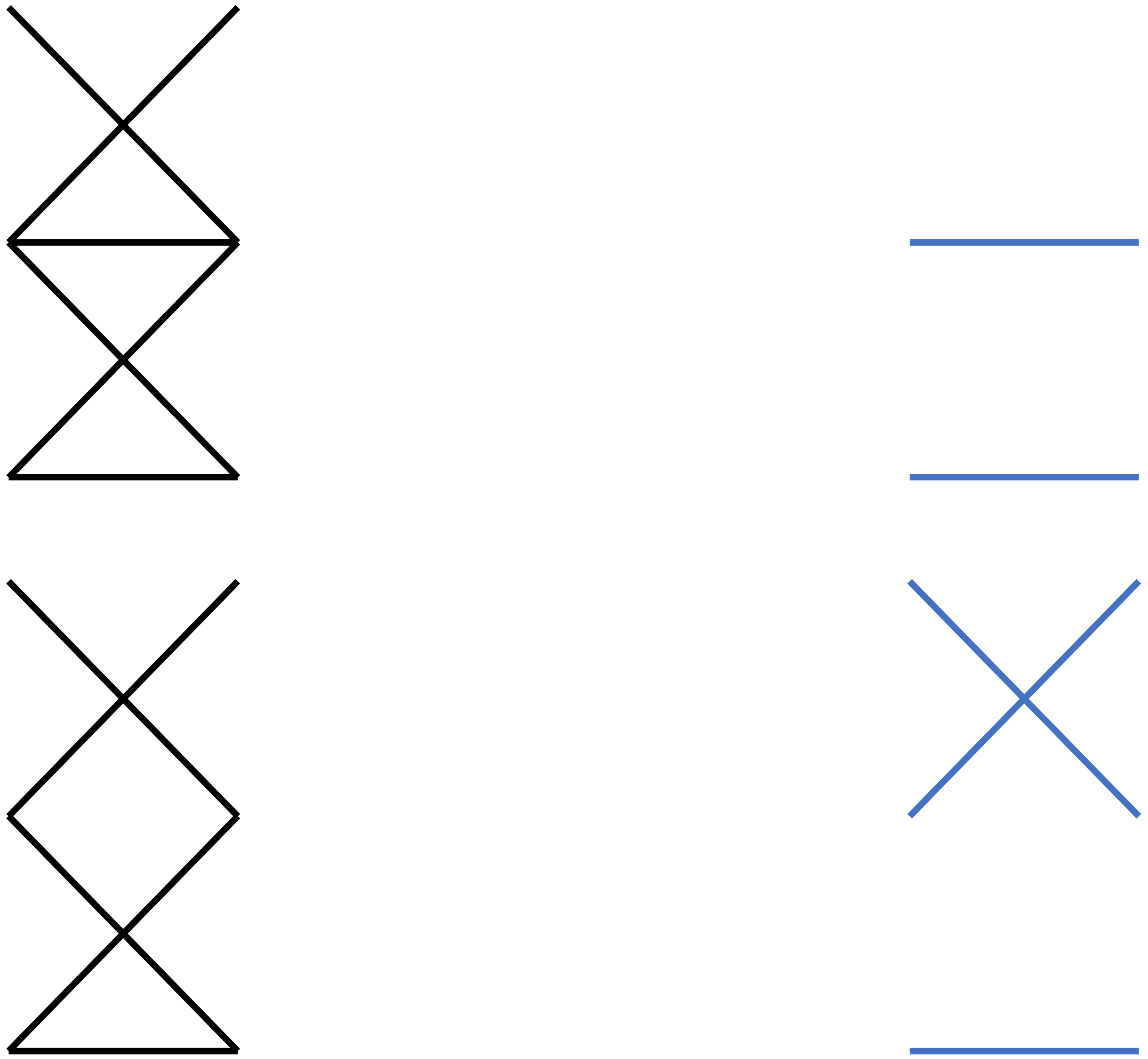

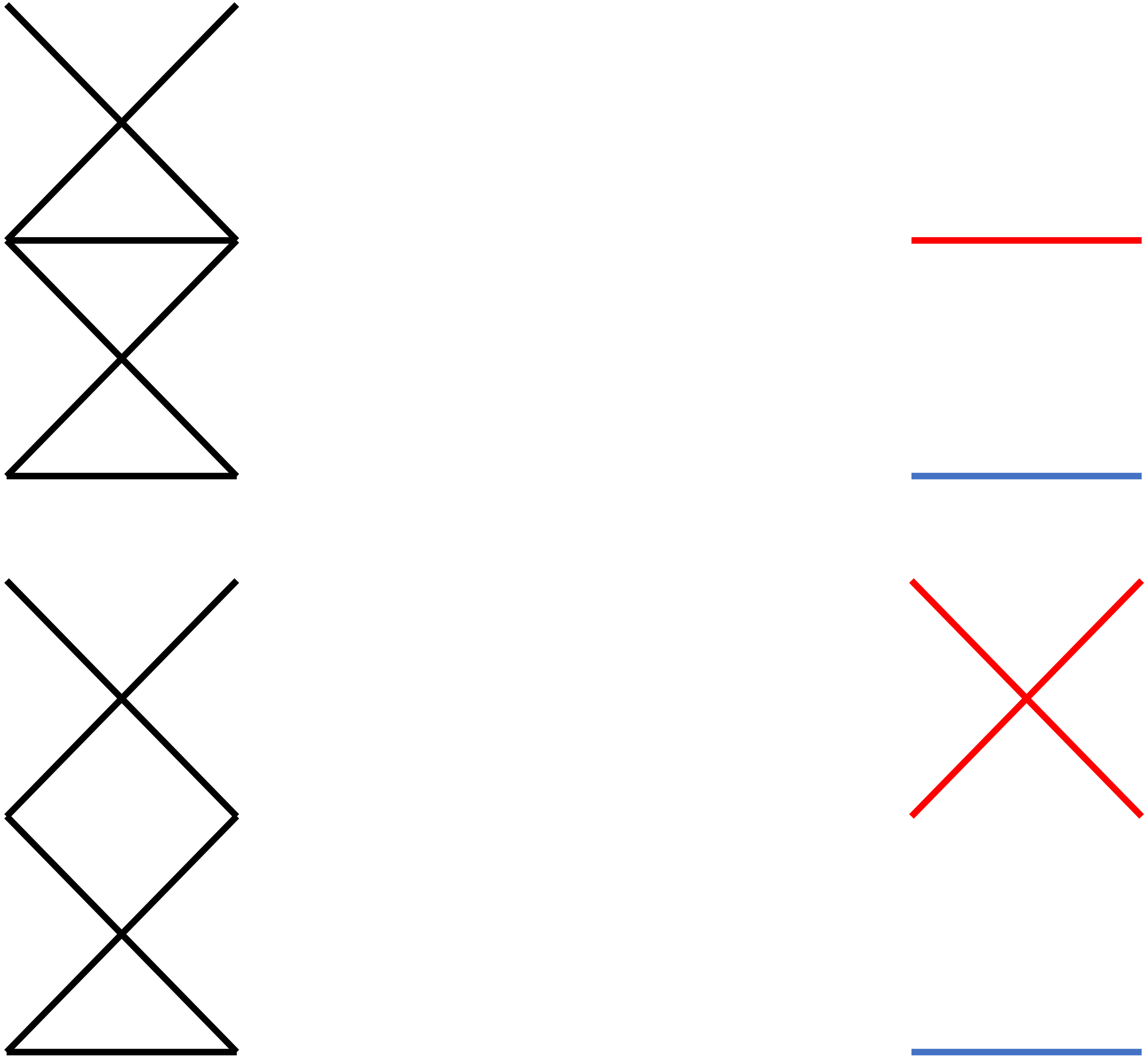

Average Sensitivity for Graph Algorithms

- Example: Greedy algorithm for maximum matching on graph with 6 nodes
- $|A(G) \triangle A(G - e)| = 3$

Average Sensitivity for Graph Algorithms

- Measure of how much a discrete algorithm changes when the input changes
- For a randomized graph algorithm A on a graph G with n vertices, the average sensitivity is defined as

$$\mathop{\mathrm{E}}_{e \sim E(G)}[d_{EMD}(A(G), A(G-e))]$$

where d_{EMD} is the Earth Mover Distance


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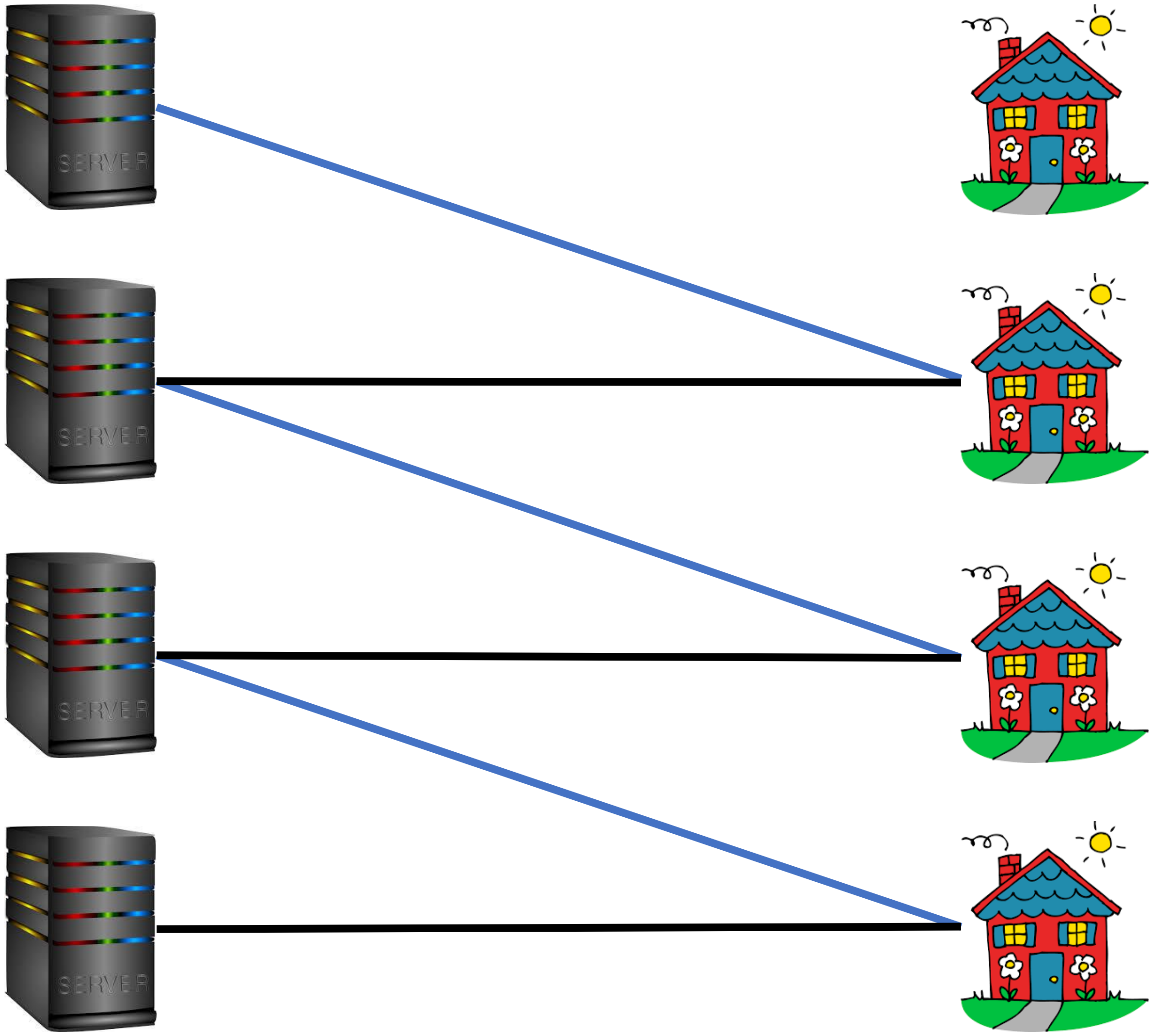
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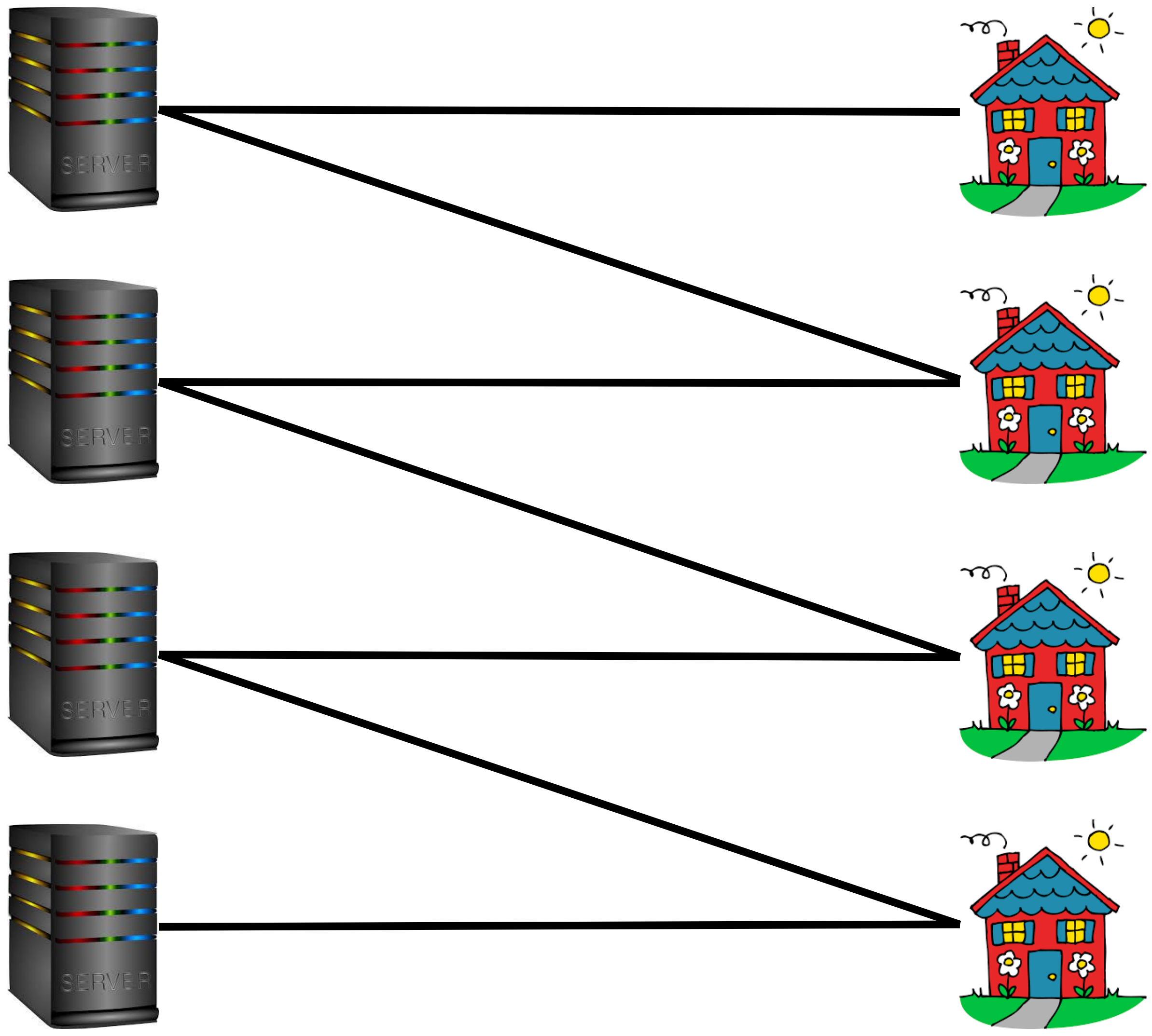
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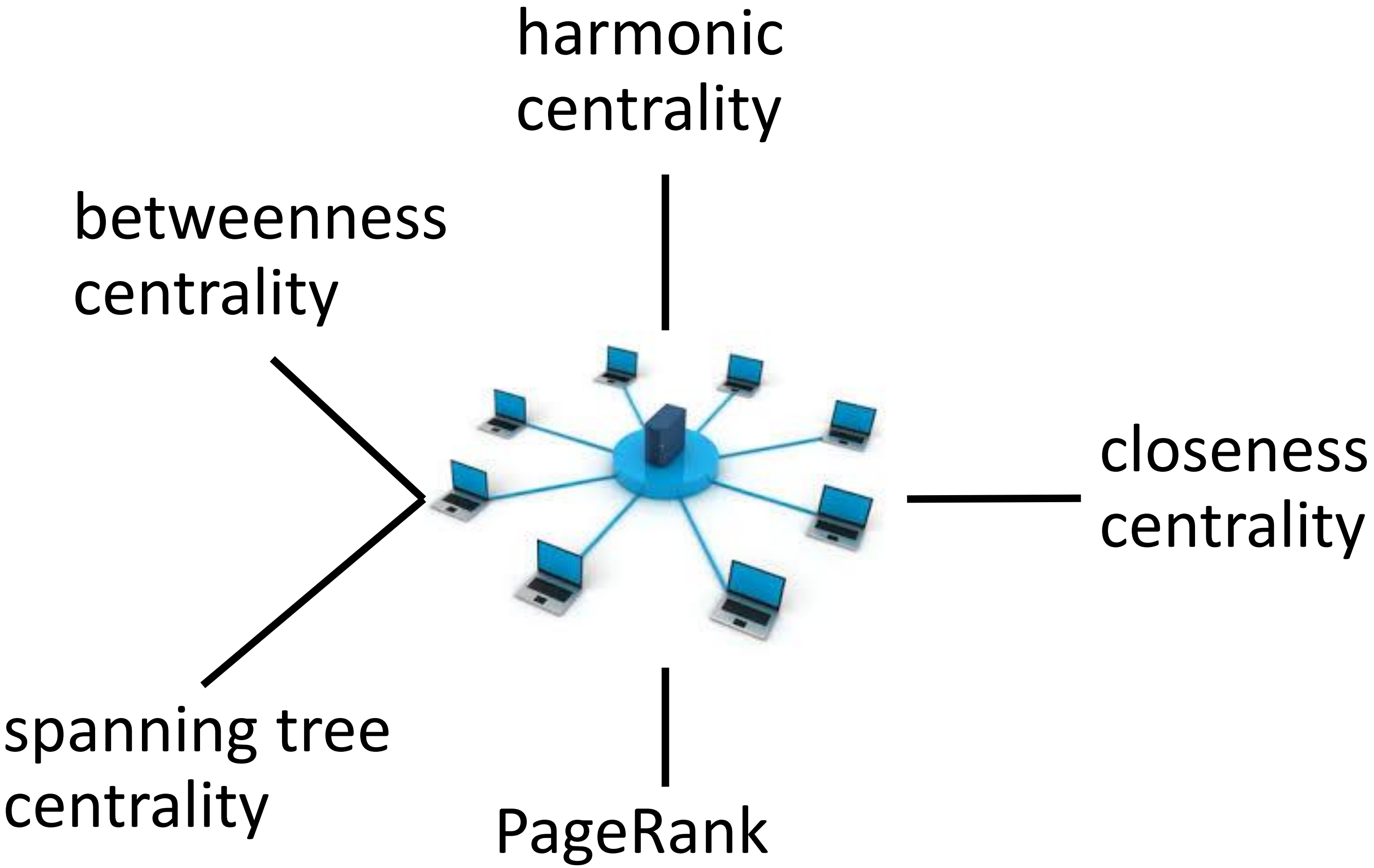
Average Sensitivity for Graph Algorithms

- Introduced by Murai and Yoshida [MY19] for centrality



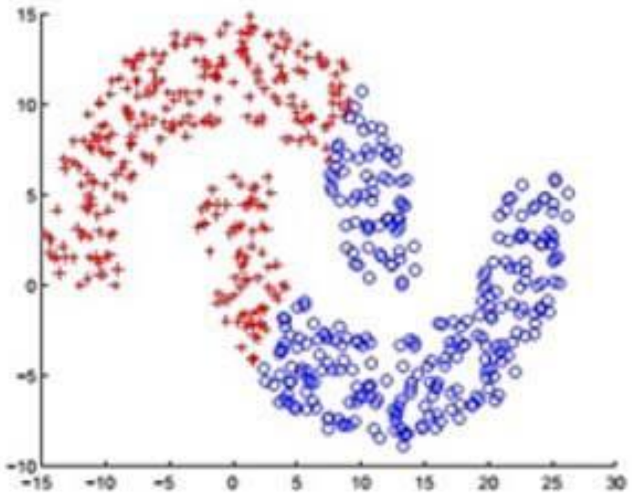
Average Sensitivity for Graph Algorithms

- Introduced by Murai and Yoshida [MY19] for centrality
- Varma and Yoshida [VY21] for graph algorithms that output edge/vertex sets

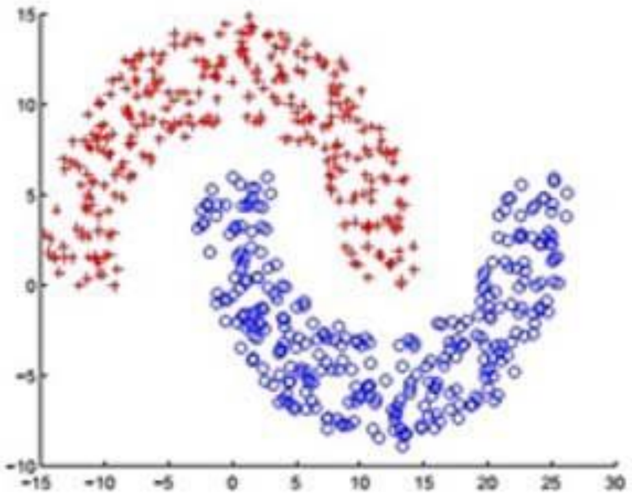
Problem	Output	Approximation Guarantee	Average Sensitivity
Minimum Spanning Forest	Edge set	1 $< \infty$	$O(\frac{n}{m})$ $\Omega(\frac{n}{m})$
Global Minimum Cut	Vertex set	$2+\epsilon$ 1 $< \infty$	$n^{O(\frac{1}{\epsilon \mathrm{OPT}})}$ $\Omega(n)$ $\Omega\left(\frac{n^{\frac{1}{\mathrm{OPT}}}}{\mathrm{OPT}^2}\right)$
Minimum s-t Cut	Vertex set	additive $O(n^{2/3})$	$O\left(n^{2/3}\right)$
Maximum Matching	Edge set	$1/2$ $1-\epsilon$ 1	1 $\tilde{O}\left(\left(\frac{\mathrm{OPT}}{\epsilon^3}\right)^{\frac{1}{1+\Omega(\epsilon^2)}}\right)$ $\Omega(n)$
Minimum Vertex Cover	Vertex set	2	2
2-Coloring	Vertex set	—	$\Omega(n)$

Average Sensitivity for Graph Algorithms

- Introduced by Murai and Yoshida [MY19] for centrality
- Varma and Yoshida [VY21] for graph algorithms that output edge/vertex sets
- Peng and Yoshida [PY20] for spectral clustering



(a) K-means



(b) Spectral Clustering

From Average Sensitivity to Worst-Case Sensitivity

- We introduce the definition of worst-case sensitivity
- For a randomized graph algorithm A on a graph G with n vertices, the average sensitivity is defined as

$$\mathop{\mathrm{E}}_{e \sim E(G)} [d_{EMD}(A(G), A(G-e))]$$

- For a randomized graph algorithm A on a graph G with n vertices, the worst-case sensitivity is defined as

$$\max_{e \sim E(G)} [d_{EMD}(A(G), A(G-e))]$$

Our Results (1)

- Theorem: $(1+\epsilon)$-approximation algorithm to the maximum matching problem with worst-case edge sensitivity $O_\epsilon(1)$.
- Theorem: Deterministic algorithm for maximal matching with worst-case sensitivity Δ^x, where $x = O(6^\Delta + \log^* n)$, for graphs with vertex degree bounded by Δ.

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Maximum Matching

- $(1+\epsilon)$-approximation algorithm for maximum matching with average sensitivity $O\left(\left(\frac{OPT}{\epsilon^3}\right)^{1/(1+\Omega(\epsilon^2))}\right)$ [VY21]

- What about worst-case sensitivity? Previous example shows greedy algorithm can have sensitivity $\Omega(n)$

Maximal Matching

- ❖ First idea: Randomized greedy for maximal matching
- ❖ The “expected” deleted/added edge appears halfway through the ordering of edges
- ❖ Intuition: Either most of the edges in the matching occur early in the ordering (dense graph) OR the deleted/added edge does not affect edges in the matching (sparse graph)

Randomized greedy for maximal matching

- Let $\pi(e)$ be the rank of e in a (fixed) random permutation
- $I_\pi(e)$:= Neighbors of e with smaller rank than e
- e is in the maximal matching M iff no edges of $I_\pi(e)$ are in M
- Consider the set S of edges that must be modified to maintain the invariant if e is moved to the beginning of π
- Unless $\pi(e) = \min \pi(S)$, then no edges need to be modified
- $\pi(e) = \min \pi(S)$ with probability $\frac{1}{|S|}$
- Expected number of modified edges $O(1)$

Randomized greedy for maximal matching

- Let π … e the rank of … (fixed) … permutation
- $I_{\pi}(e)$:= Ne…
- Consi… …intain the invariant i…
- Unless $\pi(e) =$ … odified
- $\pi(e) = \min \pi$ … with … ility $\frac{}{|S|}$
- Expected number of mo… ed edges $O(1)$

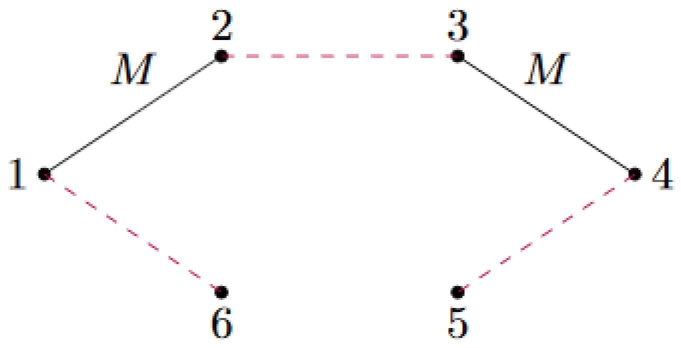
Similar idea used for maximal independent set in the dynamic distributed setting [CHK16]

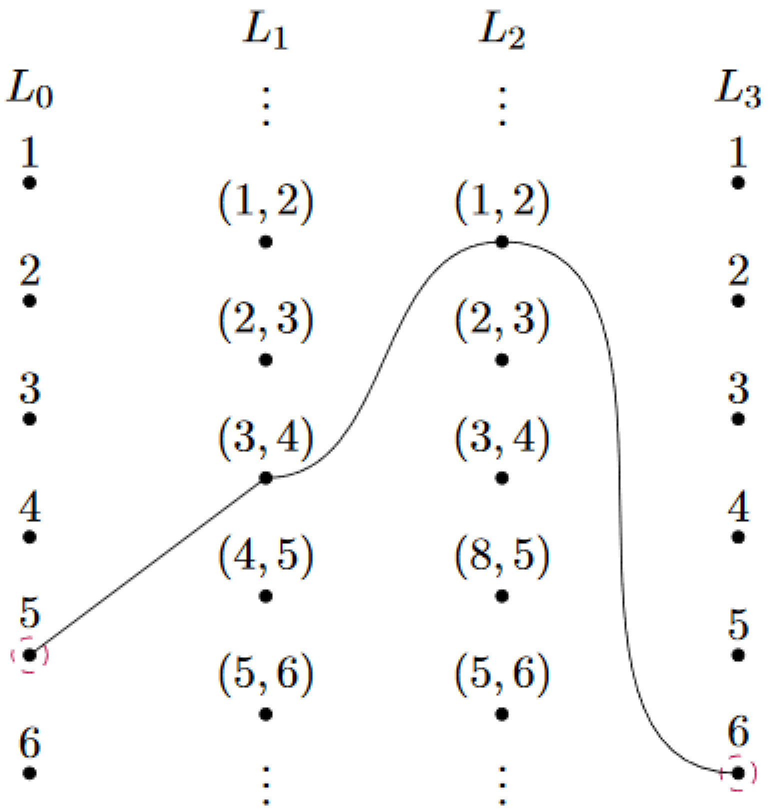
Step 1: Maximal Matching

- First idea: Randomized greedy for maximal matching with worst-case sensitivity $O(1)$
- From maximal matching to $(1+\epsilon)$-approximation algorithm for maximum matching?
- Modify the layered graph in the multi-pass streaming algorithm of McGregor [M05]

Step 2: Layered Graph

- Creates a graph with $\ell + 1$ layers to simultaneously sample a large number of augmenting paths of length ℓ





Step 2: Layered Graph

- Creates a graph with $\ell + 1$ layers to simultaneously sample a large number of augmenting paths of length ℓ
- Terminates in roughly $\frac{1}{\epsilon}$ iterations to achieve $(1+\epsilon)$-approximation algorithm for maximum matching
- Samples roughly $K = \left(\frac{1}{\epsilon}\right)^{2^{O\left(\frac{1}{\epsilon}\right)}}$ augmenting paths
- Worst-case sensitivity $O(3^K) = O_\epsilon(1)$
- Runtime: $O\big((n+m)K\big)$

Our Results (1)

- Theorem: $(1+\epsilon)$-approximation algorithm to the maximum matching problem with worst-case edge sensitivity $O_\epsilon(1)$.
- Theorem: Deterministic algorithm for maximal matching with worst-case sensitivity Δ^{x}, where $x = O(6^\Delta + \log^* n)$, for graphs with vertex degree bounded by Δ.

Deterministic Algorithm

- Ingredient #1: Deterministic local computation algorithm (LCA) by Cole and Vishkin for 6^{Δ}-coloring of a graph with degree Δ using $O(\Delta \log^* n)$ queries [CV86]
- Partitions the graph into forests and assigns each node a color based on the LSB that differs between the node ID and its parent ID
- Fact #1: Deterministic LCA only queries vertices within distance $O(\log^* n)$

Deterministic Algorithm

- Ingredient #2: Framework by Parnas and Ron [PR07] that simulates local distributed algorithms using deterministic LCA using $\Delta^{O(c)}$ probes
- Iterate through colors and add each edge to maximal matching M if no adjacent edge is already in M
- Fact #2: Framework only queries vertices within distance $O(6^{\Delta})$

Deterministic Algorithm

- Fact #1: Deterministic LCA only queries vertices within distance $O(\log^* n)$
- Fact #2: Framework only queries vertices within distance $O(6^\Delta)$
- Sensitivity analysis: At most $O\left(\Delta^{6^\Delta + \log^* n}\right)$ queries in the LCA can be affected by a single deletion/addition

Our Results (1)

- Theorem: $(1+\epsilon)$-approximation algorithm to the maximum matching problem with worst-case edge sensitivity $O_\epsilon(1)$.
- Theorem: Deterministic algorithm for maximal matching with worst-case sensitivity Δ^x, where $x = O(6^\Delta + \log^* n)$, for graphs with vertex degree bounded by Δ.

Our Results (2)

- Theorem: Any deterministic constant-factor approximation algorithm for the maximum matching problem has worst-case edge sensitivity $\Omega(\log^* n)$.
- Theorem: For a graph with edge weights bounded by $W = poly(n)$ and a trade-off parameter α, there exists an algorithm that outputs a 4α-approximation to the maximum weighted matching in $O(m \log_\alpha n)$ time. For $\alpha = 2$, the algorithm has weighted sensitivity $O(W \log n)$ and normalized weighted sensitivity $O(\log n)$. For $\alpha > 2$, the algorithm has weighted sensitivity $O(W)$ and normalized weighted sensitivity $O(1)$.

Future Directions

- Worst-case analysis of monotone submodular maximization by McMeel and Yoshida [MY20]
- Worst-case (normalized) weighted sensitivity for other graph problems?
- Constant factor approximation algorithm for maximum weighted matching with “low” worst-case sensitivity?
- Can we get $(1 + \epsilon)$-approximation algorithm for maximum weighted matching with “low” worst-case (normalized) weighted sensitivity?


thank you
danke
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ngiyabonga
teşekkür ederim
gracias
merci
obrigado
bedankt
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go raibh maith agat
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